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MODE COUPLING OF GUN TUBES CAUSED BY SPACE CURVATURE AND INITIAL TWIST

H. B. KINGSBURY
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JUNE 1989

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This report formulates the equations of motion for a gun tube with arbitrary space curvature and cross section geometry and presents results showing the effects of curvature, torsion, and cross section shape on natural frequencies and mode shapes.

Expressions for the kinetic and potential energy functions of a curved and twisted tube are formulated in terms of centerline displacements and cross section rotations. These expressions are first used to derive the equation of motion and natural boundary conditions using Hamilton's Principle. The equations of motion are then specialized for various variations of centerline curvature, torsion and cross section shape along the tube axis .

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a Rayleigh-Ritz solution for the mode shapes and natural frequencies of helical tubes with fixed-fixed ends. The effects of space curvature and torsion and of cross section out-of-roundness on frequencies and mode shapes are examined.

The natural frequencies and mode shapes of tubes are found to be significantly affected by even a slight change in values of the curvature and torsion parameters. The fundamental frequencies and the degree of coupling among the displacement variables in natural modes always increased when either curvature parameter was increased. Out-of-roundness also introduces further complexity into the mode shapes.

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LIST OF SYMBOLS

A	area
a_i, b_i, c_i, d_i	coefficients
$\vec{b}$	vector binormal
C	column matrix
co_i	coefficients
E	modulus of elasticity
F	function
G	modulus of elasticity in shear
I_{xx}, I_{yy}	area moments of inertia with respect to the x and y axes
I_{xy}	product of inertia of plane area with respect to the x and y axes
k_i, kk_i	coefficients —
l	length of rod
M_a, M_b, M_c	matrices

M_d, M_e, M_f	matrices
M_x, M_y, M_z	moments
$\hat{n}$	principal vector normal
P_i	coefficients
T	kinetic energy
$\hat{t}$	vector tangent
U	strain energy
u, v, w	displacements with respect to the x, y and z axes
V	volume
V_x, V_y, V_z	forces with respect to the x, y and z axes
W	work
δW_{nc}	virtual work done by non–conservative forces
x, y, z	rectangular coordinates
α_x	angle of rotation about the x axis
α_y	angle of rotation about the y axis
β	angle

δ	variational operator
ϵ	strain
Φ	dimensionless frequency parameter
ϕ	angle of twist
η	coefficient
κ	curvature in x–z plane
λ	torsion in y–z plane
ρ	mass density
σ	stress
Ω	natural frequency
ς	coefficient

ABSTRACT

This report formulates the equations of motion for a gun tube with arbitary space curvature and cross section geometry and presents results showing the effects of curvature, torsion, and cross section shape on natural frequencies and mode shapes.

Expressions for the kinetic and potential energy functions of a curved and twisted tube are formulated in terms of centerline displacements and cross section rotations. These expressions are first used to derive the equation of motion and natural boundary conditions using Hamilton's Principle. The equations of motion are then specialized for various variations of centerline curvature, torsion and cross section shape along the tube axis.

Next, the kinetic and potential energy functions are used to formulate a Rayleigh-Ritz solution for the mode shapes and natural frequencies of helical tubes with fixed-fixed ends. The effects of space curvature and torsion and of cross section out-of-roundness on frequencies and mode shapes are examined.

The natural frequencies and mode shapes of tubes are found to be significantly affected by even a slight change in values of the curvature and torsion parameters. The fundamental frequencies and the degree of coupling among the displacement variables in natural modes always increased when either curvature parameter was increased. Out-of-roundness also introduces further complexity into the mode shapes.

CHAPTER 1

INTRODUCTION

The work presented in this report was sponsored by the United States Army for the purpose of examining the effects of space curvature and cross section out–of–roundness on the vibration mode shapes and natural frequencies of gun tubes.

In the context of structural mechanics, the problem is that of vibrations of curved and twisted rods of arbitrary cross section. The problem of vibration of rods curved in a plane, especially circular arcs, has been the subject of a large number of investigations. It is found that this type of curvature results in mode shapes in which lateral or out–of–plane motion is coupled with twisting and in–plane motion in which in–plane axial and transverse displacements are coupled. The effect of three dimensional curvature on rod vibrations does not appear to have been extensively studied. Indeed, no paper on this subject could be found in a survey of the vibration literature of the last twenty years.

Although the vibration theory of curved rods has great importance in many engineering applications and has attracted the interest of many investigators, a completely general formulation which might illuminate unnoticed dynamic effects and interactions caused by such phenomena as initial space curvature, variable shape and area of cross section, and initial twist of cross sectional principal coordinates along the rod axis is difficult to find in the literature.

One of the seminal treatments of the derivation of equations on the motion of rods with initial curvature appears in the text on elasticity by Love [10]. Although restrictive forms of Love's equations are used directly or are re-derived by subsequent investigators for small amplitude vibration of rods curved in circular arcs, there are deficiencies in Love's work which render his strain-displacement equation unsuitable as the starting point for a general examination of the motion of rods with space curvature. In Love's work, the final state of deformation is assumed to be such that cross sections remain plane and normal to the center line of the deformed rod. By these assumptions, not only is transverse shear deformation excluded but also kinematic inconsistencies are introduced.

Love's displacement was recently re-examined by Kingsbury [8, 9] who formulated governing equations for a curved and twisted rod. Kingsbury employed a completely general set of small strain but large displacement and rotation strain-displacement equations with no a priori assumptions regarding the structural action or mode of deformation of the rod. Thus, the functional form of the displacement components was unspecified. A technical theory of rods with space curvature was then developed based on the assumption that cross sections remain plane and undeformed. A system of four coupled equilibrium equations in four displacement and rotation variables resulted.

Kingsbury's technical theory of curved rods provided the starting point for this study. When a normal mode solution for the free vibration of helical rods was attempted, however, it was found that natural frequencies could not always be obtained from the resulting characteristic equations. It was felt that this problem could be the result of terms in the governing equations which caused the stiffness matrix of the characteristic equation

to be non-symmetric. A second problem encountered was that of uncertainty concerning the proper form of the force boundary conditions for the rod. It was finally realized that approximate solutions for mode shapes and natural frequencies for a variety of boundary conditions could most easily be obtained by using the Rayleigh-Ritz method which requires the use of potential and kinetic energy functions. For these reasons, it was decided to undertake a new formulation of the theory using Hamilton's Principle or the Principle of Virtual Work for dynamical systems.

The derivation of the governing differential equations employs the strain-displacement relations introduced by Kingsbury [8, 9]. These strain-displacement equations are based on the assumptions that displacements are small and the radius of the rod cross section is small in comparison to the radius of curvature. In addition, it is assumed that the cross sections remain normal to the centroidal curve locally (at centroidal axes), or in the sense of vanishing of the average transverse shear strain. The formulation of strain energy retains the effects of extension of the centroidal curve.

In Chapter 3, Hamilton's principle is used to formulate the governing equations and natural boundary conditions for a curved rod. A system of four partial differential equations in four displacement variables is obtained.

In Chapter 4, the governing differential equations (Euler-Lagrange equations) are expressed in terms of displacements, cross section properties, centroidal axis, torsion, curvature, and their derivatives for special cases.

In Chapter 5, the strain energy function for the helical rod is employed with the

Rayleigh-Ritz method to formulate approximate solutions for mode shapes and natural frequencies for helical rods with both ends fixed (clamped–clamped) and with one end fixed and the other end free (clamped–free). Two different trial functions (admissible functions) are used. Use of both types of functions allows comparison of frequencies and mode shape results obtained from those functions.

The equations of motion of curved rods derived in this report are compared with those derived by Kingsbury [8, 9] and slight differences are noted between the two. The governing equations of a ring are then compared with those presented by Love. Chapter 6 assesses the quantitative effect of these differences by employing the simplified strain energy functions with a corresponding one presented by Love to obtain Rayleigh–Ritz solutions for natural frequencies of the out–of–plane vibrations of a clamped–clamped ring.

In Chapter 7, the Rayleigh–Ritz formulation for clamped–free helical rods shown in Chapter 5 is used to examine the effects of curvature, torsion, and out–of–roundness on the fundamental natural frequencies and mode shapes of a helical rod.

Finally, Chapter 8 briefly summarizes the conclusions drawn from the analyses performed in the course of this report.

CHAPTER 2

STRAIN ENERGY OF A GENERAL CURVED AND TWISTED ROD

In this chapter the strain energy function for a rod with general space curvature and cross section geometry is developed. The strain energy is phrased in terms of cross section displacement. rotation components. and the curvature and torsion of the rod centroidal axis.

2.1 Linearized Strain–Displacement Equations

Linearized strain–displacement equations for a general curved and twisted rod are presented in a paper by Kingsbury [8, 9]. These equations were based on the assumptions employed in the strength of material formulations for the bending and torsion of prismatic rods. It was assumed that a typical cross section would be unchanged in size and shape after the structure was deformed. and each cross section was assumed to undergo a small rotation around each of the three coordinate axes while its centroid was displaced along each of these axes: the resulting strain–displacement equations were :

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{xy} = 0 \tag{2.1}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} - x\frac{\partial \alpha_y}{\partial z} - y\frac{\partial \alpha_x}{\partial z} - \kappa\,(u - y\phi) \tag{2.2}$$

$$2\epsilon_{xz} = \frac{\partial u}{\partial z} - \alpha_y - \lambda v - \kappa w - y\,(\kappa\alpha_x - \frac{\partial \phi}{\partial z}) - x\,(\lambda\phi - \kappa\alpha_y) \tag{2.3}$$

$$2\epsilon_{yz} = \frac{\partial v}{\partial z} + \lambda u + \alpha_x + z\frac{\partial \phi}{\partial z} - y\lambda\phi \tag{2.4}$$

Equations (2.1), (2.2), (2.3) and (2.4) are called the strain–displacement equations. In Equations (2.2), (2.3) and (2.4), u, v and w are the displacement components at a point on the centroidal curve (the origin of the local x, y and z coordinates); λ and κ are the initial torsion and curvature,[1] while α_x, α_y and ϕ are small angles of rotation of the cross sections about the x, y and z axes respectively as shown in Figure 2-1

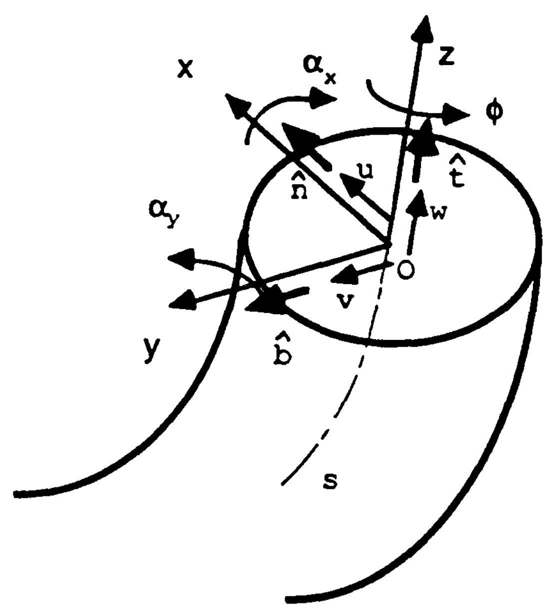

Figure 2-1: Displacement and Rotation Components

[1] The vector principal normal, binormal and tangent at point O are $\hat{n}$, $\hat{b}$, $\hat{t}$ respectively as shown in Figure 2-1. It is defined that $\frac{d\hat{t}}{ds} = \kappa$, $\frac{d\hat{b}}{ds} = -\lambda\,\hat{n}$ and $\frac{d\hat{n}}{ds} = \lambda\,\hat{b} - \kappa\,\hat{t}$, where s is the arc length.

It is noted that as in the technical theory of curved beams. the non-linear functions and the additional terms representing the warping of cross sections of non-circular rods are not considered.

When the cross sections are assumed to remain normal to the centroidal curve locally at $x = 0$ and $y = 0$. or in the average sense by integrating Equations (2.3) and (2.4) over the cross section. the equations expressing the vanishing of transverse shear deformation result in the following expressions for the cross section rotations :

$$\alpha_y = \frac{\partial u}{\partial z} - \lambda u - \kappa w \tag{2.5}$$

$$\alpha_z = -\frac{\partial v}{\partial z} - \lambda u \tag{2.6}$$

2.2 Strain Energy of a General Curved and Twisted Rod with Doubly Symmetrical Cross Section

The strain-displacement equations introduced in the last section will be used for the derivation of the strain energy function. Assuming the material constituting the rod to be linear elastic. the strain energy for the rod can be expressed by :

$$U = \frac{1}{2} \int_V \epsilon^{\mathrm{T}} \sigma \, dV \tag{2.7}$$

The stress and strain vectors in the above expression are :

$$\sigma^{\mathrm{T}} = \lfloor \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{zx} \rfloor \tag{2.8}$$

$$\epsilon^{\mathrm{T}} = \lfloor \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \rfloor \tag{2.9}$$

The applicable stress-strain relations are :

$$\sigma_{zz} = E\,\epsilon_{zz} \tag{2.10}$$

$$\tau_{xz} = 2\,G\,\epsilon_{xz} \tag{2.11}$$

$$\tau_{yz} = 2\,G\,\epsilon_{yz} \tag{2.12}$$

where E is the modulus of elasticity and G is the modulus of elasticity in shear.

Upon substituting Equations (2.1) through (2.6) and Equations (2.8) through (2.12) into Equation (2.7), the following expression for the strain energy is obtained :

$$U = \int_V P_1\,xy - P_2\,xx - P_3\,yy - P_4 - P_5\,x - P_6\,y\,dV \tag{2.13}$$

where the quantities P_1, P_2, P_3, and P_4 are presented in Appendix A.

Since the centroidal axis is the origin of the coordinate system,

$$\int_A x\,dA = 0 \tag{2.14}$$

$$\int_A y\,dA = 0 \tag{2.15}$$

P_5 and P_6 terms vanish. Hence, the strain energy is simplified to the following expression :

$$U = \int_0^l P_1\,I_{xy} - P_2\,I_{xx} - P_3\,I_{yy} - P_4\,A\;dz \tag{2.16}$$

where l is the length of the rod. According to Equation (2.16), the strain energy function per unit length of rod is given by :

$$S.E.F. = P_1\,I_{xy} - P_2\,I_{xx} - P_3\,I_{yy} - P_4\,A \tag{2.17}$$

where

$$I_{xx} = \int_A y^2 \, dA \tag{2.18}$$

$$I_{yy} = \int_A x^2 \, dA \tag{2.19}$$

$$I_{xy} = \int_A xy \, dA \tag{2.20}$$

$$A = \int_A dA \tag{2.21}$$

It is noted that the area moment inertias. area, torsion, curvature, and the tensile and shear moduli may arbitrarily vary with z. the coordinate along the curve. Although the average transverse shear strain has been set equal to zero by the use of Equations (2.5) and (2.6). extension of the centroidal curve is retained.

CHAPTER 3

DERIVATION OF THE GOVERNING EQUATIONS FOR A GENERAL CURVED AND TWISTED ROD

In this chapter the differential equations governing the deformation of the general curved and twisted rod are developed by using Hamilton's principle. For a continuous system. the use of Hamilton's principle provides an important advantage: namely. the natural boundary conditions are found systematically in the process of obtaining the equations of motion. Knowledge of the boundary conditions is necessary for determining natural frequencies by the Rayleigh-Ritz method. which will be discussed in a later chapter.

3.1 Hamilton's Principle Applied in a Curved and Twisted Rod

Hamilton's principle may be stated as :

$$\int_{t_1}^{t_2} \delta\,(T - U)\, dt - \int_{t_1}^{t_2} \delta W_{nc}\, dt = 0 \tag{3.1}$$

where

T = total kinetic energy of the system.

U = potential energy of the system, including the strain energy and the potential energy of conservative external forces.

δW_{nc} = virtual work done by nonconservative forces, including damping forces

and external forces not accounted for in U.

δ = variational operator.

The potential energy and kinetic energy functions are given by :

$$U = \int_0^l P_1 I_{xx} - P_2 I_{yy} - P_3 I_{xy} - P_4 A \; dz \tag{3.2}$$

$$T = \int_0^l \frac{1}{2}\rho \; A \left(\dot{u}^2 - \dot{v}^2 + \dot{w}^2 \right) - \left(I_{xx} - I_{yy} \right) \dot{\phi}^2 \; dz \tag{3.3}$$

where

$$(\dot{\ }) = \frac{\partial (\ )}{\partial t}$$

and ρ = mass density.

According to the Equation (3.3), the kinetic energy function is defined by :

$$K.E.F. = \frac{1}{2}\rho \; A \left(\dot{u}^2 - \dot{v}^2 - \dot{w}^2 \right) - \left(I_{xx} - I_{yy} \right) \dot{\phi}^2 \tag{3.4}$$

In Equation (3.2) and Equation (3.3), u, v, w, ϕ and their derivatives can be considered to be functions of the independent variables z and t.

For the undamped free vibration problem, there are no nonconservative or external forces acting on the system. Combining Equations (3.2) and (3.3), substituting into Equation (3.1), then integrating by parts and equating the resultant coefficients of δu, δv, δw and $\delta\phi$ yields the following Euler–Lagrange equations as well as the natural boundary conditions for the system.

$$\frac{\partial^2}{\partial z^2}\frac{\partial F}{\partial u''} - \frac{\partial}{\partial z}\frac{\partial F}{\partial u'} - \frac{\partial F}{\partial u'} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{u}} = 0 \tag{3.5}$$

$$\frac{\partial^2}{\partial z^2}\frac{\partial F}{\partial v''} - \frac{\partial}{\partial z}\frac{\partial F}{\partial v'} - \frac{\partial F}{\partial v'} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{v}} = 0 \tag{3.6}$$

$$-\frac{\partial}{\partial z}\frac{\partial F}{\partial w'} - \frac{\partial F}{\partial w} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{w}} = 0 \tag{3.7}$$

$$-\frac{\partial}{\partial z}\frac{\partial F}{\partial o'} - \frac{\partial F}{\partial o} - \frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{o}} = 0 \tag{3.8}$$

where

$$(\ )' = \frac{\partial(\ )}{\partial z}\ .$$

$$F(u, u', u'', \dot{u}, v, v', v'', \dot{v}, w, w', \dot{w}, o, o', \dot{u})$$

$$= \frac{1}{2}\rho\ \ A(\dot{u}^2 - \dot{v}^2 - \dot{w}^2) - (I_{xx} - I_{yy})\dot{o}^2$$

$$-(\mathrm{P}_1 I_{xy} - \mathrm{P}_2 I_{xx} - \mathrm{P}_3 I_{yy} - \mathrm{P}_3 A)\ . \tag{3.9}$$

and at $z = 0$ and $z = l$

Either u' *is specified or* $\frac{\partial F}{\partial u''} = 0$

Either u *is specified or* $\frac{\partial}{\partial z}\frac{\partial F}{\partial u''} - \frac{\partial F}{\partial u'} = 0$

Either v' *is specified or* $\frac{\partial F}{\partial v''} = 0$

Either v *is specified or* $\frac{\partial}{\partial z}\frac{\partial F}{\partial v''} - \frac{\partial F}{\partial v'} = 0$

Either w *is specified or* $\frac{\partial F}{\partial w'} = 0$

Either o *is specified or* $\frac{\partial F}{\partial o'} = 0$

CHAPTER 4

EXPRESSION OF EULER–LAGRANGE EQUATIONS FOR GENERAL AND SPECIAL CASES

In this chapter, the Euler–Lagrange equations for a rod with space curvature are expressed in terms of the rod displacement functions, torsion, curvature, section properties, and their derivatives. The general equations are then specialized for various cases of centerline curvature.

4.1 General Form

Upon substituting the function F, Equation (3.9), into Equations (3.5), (3.6), (3.7) and (3.8), the following equations are obtained :

$$EI_{yy}\kappa w'''$$

$$-(3EI_{yy}\kappa' - (2EI_{yy}' - E\lambda I_{xy})\kappa)w''$$

$$-(3EI_{yy}\kappa'' - (4EI_{yy}' - 2E\lambda I_{xy})\kappa' - GI_{yy}\kappa^3 - (EI_{yy}'' - E\lambda I_{xy}' - AE)\kappa)w'$$

$$-(EI_{yy}\kappa''' - (2EI_{yy}' - E\lambda I_{xy})\kappa''$$

$$-(-3GI_{yy}\kappa^2 - EI_{yy}'' - E\lambda I_{xy}')\kappa' - (G\lambda I_{xy} - GI_{yy}')\kappa^3)w$$

$$-EI_{xy}v''''$$

$$-(-E\lambda I_{yy} - 2EI_{xy}' - E\lambda I_{xx})v'''$$

$$-(-GI_{xy}\kappa^2-2E\lambda I_{yy}{}'-3E\lambda' I_{yy}-EI_{xy}{}''-E\lambda^2 I_{xy}-E\lambda I_{xx}{}')v''$$

$$-(-2GI_{xy}\kappa\kappa'-(G\lambda I_{yy}-GI_{xy}{}'-G\lambda I_{xx})\kappa^2$$

$$-E\lambda I_{yy}{}''-4E\lambda' I_{yy}{}'-3E\lambda'' I_{yy}-E\lambda^2 I_{xy}{}'-2E\lambda\lambda' I_{xy})v'$$

$$-(2G\lambda I_{yy}\kappa\kappa'-(G\lambda I_{yy}{}'-G\lambda' I_{yy}-G\lambda^2 I_{xy})\kappa^2$$

$$-E\lambda' I_{yy}{}''-2E\lambda'' I_{yy}{}'-E\lambda''' I_{yy}-E\lambda\lambda' I_{xy}{}'-E\lambda\lambda'' I_{xy})v$$

$$-EI_{yy}u''''$$

$$-2EI_{yy}{}'u'''$$

$$-(-GI_{yy}\kappa^2-EI_{yy}{}''-E\lambda I_{xy}{}'-3E\lambda' I_{xy}-E\lambda^2 I_{xx})u''$$

$$-(-2GI_{yy}\kappa\kappa'-GI_{yy}{}'\kappa^2$$

$$-E\lambda I_{xy}{}''-4E\lambda' I_{xy}{}'-3E\lambda'' I_{xy}-E\lambda^2 I_{xx}{}'-2E\lambda\lambda' I_{xx})u'$$

$$-(-2G\lambda I_{xy}\kappa\kappa'-(-G\lambda I_{xy}{}'-G\lambda' I_{xy}-G\lambda^2 I_{xx}-AE)\kappa^2-E\lambda' I_{xy}{}''$$

$$-2E\lambda'' I_{xy}{}'-E\lambda''' I_{xy}-E\lambda\lambda' I_{xx}{}'-E\lambda\lambda'' I_{xx})u$$

$$-(-G-E)I_{xy}\kappa\phi''$$

$$-((-G-2E)I_{xy}\kappa'-(-G\lambda I_{yy}-(-G-2E)I_{xy}{}'-(G-E)\lambda I_{xx})\kappa)\phi'$$

$$-(-EI_{xy}\kappa''-(-G\lambda I_{yy}-2EI_{xy}{}'-E\lambda I_{xx})\kappa'$$

$$-(-G\lambda I_{yy}{}'-G\lambda' I_{yy}-EI_{xy}{}''+G\lambda^2 I_{xy}+E\lambda I_{xx}{}')\kappa)\phi = -A\rho\ddot{u} \qquad (4.1)$$

$$EI_{xy}\kappa w'''$$

$$-(3EI_{xy}\kappa'-(E\lambda I_{yy}-2EI_{xy}{}')\kappa)w''$$

$$-(3EI_{xy}\kappa'' - (2E\lambda I_{yy} - 4EI_{xy}')\kappa' - GI_{xy}\kappa^3 - (E\lambda I_{yy}' - EI_{xy}'')\kappa)w'$$

$$-(EI_{xy}\kappa''' - (E\lambda I_{yy} - 2EI_{xy}')\kappa''$$

$$-(-3GI_{xy}\kappa^2 - E\lambda I_{yy}' - EI_{xy}'')\kappa' - (-G\lambda I_{yy} - GI_{xy}')\kappa^3)w$$

$$-EI_{xx}v''''$$

$$-2EI_{xx}'v'''$$

$$-(-GI_{xx}\kappa^2 - E\lambda^2 I_{yy} - E\lambda I_{xy}' - 3E\lambda' I_{xy} - EI_{xx}'')v''$$

$$-(-2GI_{xx}\kappa\kappa' - GI_{xx}'\kappa^2 - E\lambda^2 I_{yy}' - 2E\lambda\lambda' I_{yy}$$

$$-E\lambda I_{xy}'' - 4E\lambda' I_{xy}' - 3E\lambda'' I_{xy})v'$$

$$-(2G\lambda I_{xy}\kappa\kappa' - (G\lambda^2 I_{yy} - G\lambda I_{xy}' - G\lambda' I_{xy})\kappa^2 - E\lambda\lambda' I_{yy}'$$

$$-E\lambda\lambda'' I_{yy} - E\lambda' I_{xy}'' - 2E\lambda'' I_{xy}' - E\lambda''' I_{xy})v$$

$$-EI_{xy}u''''$$

$$-(E\lambda I_{yy} - 2EI_{xy}' - E\lambda I_{xx})u'''$$

$$-(-GI_{xy}\kappa^2 - E\lambda I_{yy}' - EI_{xy}'' - E\lambda^2 I_{xy} - 2E\lambda I_{xx}' + 3E\lambda' I_{xx})u''$$

$$-(-2GI_{xy}\kappa\kappa' - (-G\lambda I_{yy} - GI_{xy}' - G\lambda I_{xx})\kappa^2 - E\lambda^2 I_{xy}'$$

$$-2E\lambda\lambda' I_{xy} - E\lambda I_{xx}'' - 4E\lambda' I_{xx}' - 3E\lambda'' I_{xx})u'$$

$$-(-2G\lambda I_{xx}\kappa\kappa' - (-G\lambda^2 I_{xy} - G\lambda I_{xx}' - G\lambda' I_{xx})\kappa^2 - E\lambda\lambda' I_{xy}'$$

$$-E\lambda\lambda'' I_{xy} - E\lambda' I_{xx}'' - 2E\lambda'' I_{xx}' - E\lambda''' I_{xx})u$$

$$-(-G-E)I_{xx}\kappa o''$$

$$-((-G-2E)I_{xx}\kappa'-((-2G-E)\lambda I_{xy}-(-G-2E)I_{xx}')\kappa)\phi'$$

$$-(-EI_{xx}\kappa''-((-G-E)\lambda I_{xy}-2EI_{xx}')\kappa'$$

$$-(-G\lambda^2 I_{yy}-(-G-E)\lambda I_{xy}'-G\lambda' I_{xy}-EI_{xx}'')\kappa)\phi = -A\rho\ddot{v} \tag{4.2}$$

$$(-EI_{yy}\kappa^2-AE)w''$$

$$-(-2EI_{yy}\kappa\kappa'-EI_{yy}'\kappa^2-A'E)w'$$

$$-(-EI_{yy}\kappa\kappa''-EI_{yy}'\kappa\kappa'-GI_{yy}\kappa^4)w$$

$$-EI_{xy}\kappa v'''$$

$$-(E\lambda I_{yy}-EI_{xy}')\kappa v''$$

$$-(GI_{xy}\kappa^3-(E\lambda I_{yy}'-2E\lambda' I_{yy})\kappa)v'$$

$$-((E\lambda' I_{yy}'-E\lambda'' I_{yy})\kappa-G\lambda I_{yy}\kappa^3)v$$

$$-EI_{yy}\kappa u'''$$

$$-(-EI_{yy}'-E\lambda I_{xy})\kappa u''$$

$$-(GI_{yy}\kappa^3-(-E\lambda I_{xy}'-2E\lambda' I_{xy}-AE)\kappa)u'$$

$$-(AE\kappa'-G\lambda I_{xy}\kappa^3-(-E\lambda' I_{xy}'-E\lambda'' I_{xy}-A'E)\kappa)u$$

$$-(G-E)I_{xy}\kappa^2\phi'$$

$$-(EI_{xy}\kappa\kappa'-(G\lambda I_{yy}-EI_{xy}')\kappa^2)\phi = -A\rho\ddot{w} \tag{4.3}$$

$$(-G-E)I_{xy}\kappa^2 w'$$

$$-((-2G-E)I_{xy}\kappa\kappa' - (G\lambda I_{yy} - GI_{xy}')\kappa^2)w$$

$$-(-G-E)I_{xx}\kappa v''$$

$$-(((2G-E)\lambda I_{xy} - GI_{xx}')\kappa - GI_{xx}\kappa')v'$$

$$-(G\lambda I_{xy}\kappa' - (-G\lambda^2 I_{yy} - G\lambda I_{xy}' - (G-E)\lambda' I_{xy})\kappa)v$$

$$-(-G-E)I_{xy}\kappa u''$$

$$-((G\lambda I_{yy} - GI_{xy}' - (-G-E)\lambda I_{xx})\kappa - GI_{xy}\kappa')u'$$

$$-((G\lambda^2 I_{xy} - G\lambda I_{xx}' - (-G-E)\lambda' I_{xx})\kappa - G\lambda I_{xx}\kappa')u$$

$$-(-GI_{yy} - GI_{xx})o''$$

$$-(-GI_{yy}' - GI_{xx}')o'$$

$$-(EI_{xx}\kappa^2 - G\lambda^2 I_{yy} - G\lambda^2 I_{xx})o = -(I_{yy} - I_{xx})\ddot{o}\rho \tag{4.4}$$

The effects of torsion, curvature, and the area moment of inertia of the cross section, as well as the variation of these quantities along the rod, are next examined. When these equations are compared with the equilibrium equations derived by Kingsbury [8, 9] and others for various special cases, certain differences are observed. The effects of these differences for the case of a circular ring are examined in Chapter 6.

4.2 Straight Rod with Variable Section Properties

In the case of a straight rod. κ and λ and their derivatives are equal to zero: thus. Equations (4.1). (4.2). (4.3) and (4.4) are simplified to :

$$EI_{xy}v'''' + 2EI_{xy}'v''' + EI_{xy}''v'' + EI_{yy}u'''' + 2EI_{yy}'u''' + EI_{yy}''u''$$

$$= - A\rho \ddot{u} \tag{4.5}$$

$$EI_{xx}v'''' + 2EI_{xx}'v''' + EI_{xx}''v'' + EI_{xy}u'''' + 2EI_{xy}'u''' + EI_{xy}''u''$$

$$= - A\rho \ddot{v} \tag{4.6}$$

$$- AEw'' - A'Ew' = - A\rho \ddot{w} \tag{4.7}$$

$$(- GI_{yy} - GI_{xx})o'' + (- GI_{yy}' - GI_{xx}')o' = - (I_{yy} + I_{xx})\rho \ddot{o} \tag{4.8}$$

From the above equations. it is seen that if the area moment of inertia (I_{xy}) is equal to zero. then u and v are uncoupled.

4.3 Rod Curved in a Plane with Constant Curvature and Variable Section Properties

In this case. the rod is initially in the form of a segment of a circle so that κ is a constant and λ is zero. The Euler-Lagrange equations are thus simplified to :

$$EI_{yy}\kappa w''' + 2EI_{yy}'\kappa w'' + ((EI_{yy}'' - AE)\kappa - GI_{yy}\kappa^3)w'$$

$$- GI_{yy}'\kappa^3 w + EI_{xy}v'''' + 2EI_{xy}'v''' + (EI_{xy}'' - GI_{xy}\kappa^2)v'' - GI_{xy}'\kappa^2 v'$$

$$+ EI_{yy}u'''' + 2EI_{yy}'u''' + (EI_{yy}'' - GI_{yy}\kappa^2)u'' - GI_{yy}'\kappa^2 u' + AE\kappa^2 u$$

$$-(-G-E)I_{xy}\kappa\phi'' - (-G-2E)I_{xy}'\kappa\phi' - EI_{xy}''\kappa\phi = -A\rho\ddot{u} \tag{4.9}$$

$$EI_{xy}\kappa w''' - 2EI_{xy}'\kappa w'' - (EI_{xy}''\kappa - GI_{xy}\kappa^3)w' - GI_{xy}'\kappa^3 w$$

$$-EI_{xx}v'''' - 2EI_{xx}'v''' - (EI_{xx}'' - GI_{xx}\kappa^2)v'' - GI_{xx}'\kappa^2 v'$$

$$-EI_{xy}u'''' - 2EI_{xy}'u''' - (EI_{xy}'' - GI_{xy}\kappa^2)u'' - GI_{xy}'\kappa^2 u'$$

$$-(-G-E)I_{xx}\kappa\phi'' - (-G-2E)I_{xx}'\kappa\phi' - EI_{xx}''\kappa\phi = -A\rho\ddot{v} \tag{4.10}$$

$$(-EI_{yy}\kappa^2 - AE)w'' - (-EI_{yy}'\kappa^2 - A'E)w' - GI_{yy}\kappa^4 w$$

$$-EI_{xy}\kappa v''' - EI_{xy}'\kappa v'' - GI_{xy}\kappa^3 v'$$

$$-EI_{yy}\kappa u''' - EI_{yy}'\kappa u'' - (GI_{yy}\kappa^3 - AE\kappa)u' - A'E\kappa u$$

$$-(G-E)I_{xy}\kappa^2\phi' - EI_{xy}'\kappa^2\phi = -A\rho\ddot{w} \tag{4.11}$$

$$(-G-E)I_{xy}\kappa^2 w' - GI_{xy}'\kappa^2 w$$

$$-(-G-E)I_{xx}\kappa v'' - GI_{xx}'\kappa v'$$

$$-(-G-E)I_{xy}\kappa u'' - GI_{xy}'\kappa u'$$

$$-(-GI_{yy} - GI_{xx})\phi'' - (-GI_{yy}' - GI_{xx}')\phi' - EI_{xx}\kappa^2\phi = -(I_{yy} - I_{xx})\rho\ddot{\phi} \tag{4.12}$$

The above equations are fully coupled in the variables. However, if I_{xy} is zero as in the case of. for example. a circular cross section. then two pairs of equations result.

4.4 Rod Curved in a Plane with Variable Curvature but Constant Section Properties

The rod curved in a plane with variable curvature but constant section properties implies that the λ and its derivatives are zero and that the derivatives of the section properties are zero also. Substituting these values into the general form Equations, (4.1) (4.2) (4.3) and (4.4), yields :

$$EI_{yy}\kappa w''' - 3EI_{yy}\kappa' w'' - (3EI_{yy}\kappa'' - GI_{yy}\kappa^3 - AE\kappa)w'$$

$$-(EI_{yy}\kappa''' - 3GI_{yy}\kappa^2\kappa')w$$

$$-EI_{xy}v'''' - GI_{xy}\kappa^2 v'' - 2GI_{xy}\kappa\kappa' v'$$

$$-EI_{yy}u'''' - GI_{yy}\kappa^2 u'' - 2GI_{yy}\kappa\kappa' u' - AE\kappa^2 u$$

$$-(-G-E)I_{xy}\kappa\phi'' - (-G-2E)I_{xy}\kappa'\phi' - EI_{xy}\kappa''\phi = -A\rho\ddot{u} \tag{4.13}$$

$$EI_{xy}\kappa w''' - 3EI_{xy}\kappa' w'' - (3EI_{xy}\kappa'' - GI_{xy}\kappa^3)w' - (EI_{xy}\kappa''' - 3GI_{xy}\kappa^2\kappa')w$$

$$-EI_{xx}v'''' - GI_{xx}\kappa^2 v'' - 2GI_{xx}\kappa\kappa' v'$$

$$-EI_{xy}u'''' - GI_{xy}\kappa^2 u'' - 2GI_{xy}\kappa\kappa' u'$$

$$-(-G-E)I_{xx}\kappa\phi'' - (-G-2E)I_{xx}\kappa'\phi' - EI_{xx}\kappa''\phi = -A\rho\ddot{v} \tag{4.14}$$

$$(-EI_{yy}\kappa^2 - AE)w'' - 2EI_{yy}\kappa\kappa' w' + (-EI_{yy}\kappa\kappa'' + GI_{yy}\kappa^4)w$$

$$-EI_{xy}\kappa v''' - GI_{xy}\kappa^2 v'$$

$$-EI_{yy}\kappa u''' - (GI_{yy}\kappa^3 - AE\kappa)u' + AE\kappa' u$$

$$-(G-E)I_{xy}\kappa^2\phi' - EI_{xy}\kappa\kappa'\phi = -A\rho\ddot{w} \tag{4.15}$$

$$(-G-E)I_{xy}\kappa^2 w' - (-2G-E)I_{xy}\kappa\kappa' w$$

$$-(-G-E)I_{xx}\kappa v'' - GI_{xx}\kappa' v'$$

$$-(-G-E)I_{xy}\kappa u'' - GI_{xy}\kappa' u'$$

$$-(-GI_{yy}-GI_{xx})\phi'' - EI_{xx}\kappa^2\phi = -(I_{yy}-I_{xx})\rho\ddot{\phi} \tag{4.16}$$

If I_{xy} is set equal to zero in the above equations, the equations governing in-plane motions Eq. (4.13) and Eq. (4.15) uncouple from those of out-of-plane motions Eq. (4.14) and Eq. (4.16).

4.5 Rod with Constant Curvature and Torsion and with Cross Section Varying along the Centroidal Line and Symmetric to the Local Origin

In this special case, the rod axis is a helix and I_{xy} is zero. The governing equations become :

$$EI_{yy}\kappa w'''$$

$$-(2EI_{yy}' - E\lambda I_{xy})\kappa w''$$

$$-((EI_{yy}'' - E\lambda I_{xy}' - AE)\kappa - GI_{yy}\kappa^3)w'$$

$$-(G\lambda I_{xy} - GI_{yy}')\kappa^3 w$$

$$-EI_{xy}v''''$$

$$-(-E\lambda I_{yy} - 2EI_{xy}' - E\lambda I_{xx})v'''$$

$$-(-GI_{xy}\kappa^2-2E\lambda I_{yy}{}'-EI_{xy}{}''-E\lambda^2 I_{xy}-E\lambda I_{xx}{}')v''$$

$$-((G\lambda I_{yy}-GI_{xy}{}'-G\lambda I_{xx})\kappa^2-E\lambda I_{yy}{}''-E\lambda^2 I_{xy}{}')v'$$

$$-(G\lambda I_{yy}{}'-G\lambda^2 I_{xy})\kappa^2 v$$

$$-EI_{yy}u''''$$

$$-2EI_{yy}{}'u'''$$

$$-(-GI_{yy}\kappa^2-EI_{yy}{}''-E\lambda I_{xy}{}'-E\lambda^2 I_{xx})u''$$

$$-(-GI_{yy}{}'\kappa^2-E\lambda I_{xy}{}''-E\lambda^2 I_{xx}{}')u'$$

$$-(-G\lambda I_{xy}{}'-G\lambda^2 I_{xx}-AE)\kappa^2 u$$

$$-(-G-E)I_{xy}\kappa o''$$

$$-(-G\lambda I_{yy}-(-G-2E)I_{xy}{}'-(G-E)\lambda I_{xx})\kappa o'$$

$$-(-G\lambda I_{yy}{}'-EI_{xy}{}''-G\lambda^2 I_{xy}-E\lambda I_{xx}{}')\kappa o = -A\rho \ddot{u} \tag{4.17}$$

$$EI_{xy}\kappa w'''$$

$$-(E\lambda I_{yy}-2EI_{xy}{}')\kappa w''$$

$$-((E\lambda I_{yy}{}'-EI_{xy}{}'')\kappa-GI_{xy}\kappa^3)w'$$

$$-(-G\lambda I_{yy}-GI_{xy}{}')\kappa^3 w$$

$$-EI_{xx}v''''$$

$$-2EI_{xx}{}'v'''$$

$$-(-GI_{xx}\kappa^2-E\lambda^2 I_{yy}-E\lambda I_{xy}{}'-EI_{xx}{}'')v''$$

$$-(-GI_{xx}{}'\kappa^2 - E\lambda^2 I_{yy}{}' - E\lambda I_{xy}{}'')v'$$

$$-(G\lambda^2 I_{yy} - G\lambda I_{xy}{}')\kappa^2 v$$

$$-EI_{xy}u''''$$

$$-(E\lambda I_{yy} - 2EI_{xy}{}' - E\lambda I_{xx})u'''$$

$$-(-GI_{xy}\kappa^2 - E\lambda I_{yy}{}' - EI_{xy}{}'' - E\lambda^2 I_{xy} - 2E\lambda I_{xx}{}')u''$$

$$-((-G\lambda I_{yy} - GI_{xy}{}' - G\lambda I_{xx})\kappa^2 - E\lambda^2 I_{xy}{}' - E\lambda I_{xx}{}'')u'$$

$$-(-G\lambda^2 I_{xy} - G\lambda I_{xx}{}')\kappa^2 u$$

$$-(-G-E)I_{xx}\kappa o''$$

$$-((-2G-E)\lambda I_{xy} - (-G-2E)I_{xx}{}')\kappa o'$$

$$-(-G\lambda^2 I_{yy} - (-G-E)\lambda I_{xy}{}' - EI_{xx}{}'')\kappa o = -A\rho\ddot{u} \tag{4.18}$$

$$(-EI_{yy}\kappa^2 - AE)w'' - (-EI_{yy}{}'\kappa^2 - A'E)w' - GI_{yy}\kappa^4 w$$

$$-EI_{xy}\kappa v''' - (E\lambda I_{yy} - EI_{xy}{}')\kappa v'' - (GI_{xy}\kappa^3 - E\lambda I_{yy}{}'\kappa)v' - G\lambda I_{yy}\kappa^3 v$$

$$-EI_{yy}\kappa u''' - (-EI_{yy}{}' - E\lambda I_{xy})\kappa u''$$

$$-(GI_{yy}\kappa^3 - (AE - E\lambda I_{xy}{}')\kappa)u' - (G\lambda I_{xy}\kappa^3 - A'E\kappa)u$$

$$-(G-E)I_{xy}\kappa^2 o' - (G\lambda I_{yy} - EI_{xy}{}')\kappa^2 o = -A\rho\ddot{w} \tag{4.19}$$

$$(-G-E)I_{xy}\kappa^2 w' - (G\lambda I_{yy} - GI_{xy}{}')\kappa^2 w$$

$$-(-G-E)I_{xx}\kappa v'' - ((2G-E)\lambda I_{xy} - GI_{xx}{}')\kappa v' - (G\lambda I_{xy}{}' - G\lambda^2 I_{yy})\kappa v$$

$$-(-G-E)I_{xy}\kappa u''-(G\lambda I_{yy}-GI_{xy}'-(-G-E)\lambda I_{xx})\kappa u'-(G\lambda^2 I_{xy}-G\lambda I_{xx}')\kappa u$$

$$-(-GI_{yy}-GI_{xx})\phi''$$

$$-(-GI_{yy}'-GI_{xx}')\phi'-(EI_{xx}\kappa^2-G\lambda^2 I_{yy}-G\lambda^2 I_{xx})\phi = -(I_{yy}-I_{xx})\rho\ddot{\phi} \tag{4.20}$$

CHAPTER 5

RAYLEIGH-RITZ SOLUTION FOR THE HELICAL ROD

In this chapter, the general form of the governing equations is specialized for the helical rod which has constant torsion and curvature. The strain energy function for the helical rod is then employed with the Rayleigh–Ritz method to formulate approximate solutions for mode shapes and natural frequencies both for a helical rod with both ends fixed (clamped-clamped) and for another with one end fixed and the other free (clamped-free). For the first of these cases two different sets of trial functions are used. One of these is the set employed by Volterra and Morell [17] and by Den Hartog [5] for partial rings with clamped ends, while the second set employs the normal mode functions for clamped–clamped straight beams. Use of both types of functions allows comparison of frequencies and mode results obtained from the two different sets of trial functions. This comparison enables us to estimate the accuracy of the Rayleigh–Ritz solution for the clamped–free case which employs only the normal mode functions of straight rods (admissible functions).

5.1 Strain Energy Function for the Helical Rod

The following expression for the strain energy function for a rod with constant curvature and torsion is obtained from the general strain energy expression derived in Chapter 2.

$$
\begin{aligned}
&S.E.F._{Helical\,Rod} = \\
&((E v''^2 - E\lambda^2 u'^2 - G\kappa^2 v'^2 - G\kappa^2\lambda^2 u^2 - G\phi'^2 + G\lambda^2\phi^2 + E\kappa^2\phi^2)\ 2 \\
&- G\kappa^2\lambda u v' - G\kappa\phi' v' - E\lambda u' v'' - E\kappa\phi v'' - E\kappa\phi\lambda u' - G\kappa\lambda\phi' u) I_{xx} \\
&-(E\kappa w' v'' - E\lambda v' v'' - E u'' v'' - E\kappa\lambda w' u' - E\lambda^2 u' v' - E\lambda u' u'' \\
&- E\kappa^2\phi w' - E\kappa\phi\lambda v' - E\kappa\phi u'' - G\kappa^3 v' w - G\kappa^3\lambda u w - G\kappa^2\phi' w \\
&- G\kappa^2\lambda v v' - G\kappa^2 u' v' - G\kappa\lambda\phi v' - G\kappa^2\lambda^2 u v - G\kappa\lambda\phi' v \\
&- G\kappa^2\lambda u u' - G\kappa\phi' u' - G\kappa\lambda^2\phi u) I_{xy} \\
&-((E u''^2 - E\lambda^2 v'^2 - E\kappa^2 w'^2 - G\kappa^4 w^2 - G\kappa^2 u'^2 \\
&- G\kappa^2\lambda^2 v^2 - G\phi'^2 - G\lambda^2\phi^2)\ 2 \\
&- E\kappa w'\lambda v' - G\kappa^3\lambda v w - G\kappa^3 u' w - G\kappa^2\lambda\phi w - E\lambda v' u'' - E\kappa w' u'' \\
&- G\kappa^2\lambda u' v - G\kappa\lambda^2\phi v - G\kappa\lambda\phi u') I_{yy} \\
&-((E w'^2 - E\kappa^2 u^2)\ 2 - E\kappa u w') A
\end{aligned}
\tag{5.1}
$$

Although the section properties in the above strain energy function remain functions of z, in the subsequent sections of this chapter these section properties will be kept as constant.

5.2 Free Vibrations of a Clamped–Clamped Helical Rod with Constant Section Properties

With reference to Section 3.1, the natural boundary conditions for the clamped–clamped case are $u' = 0$, $u = 0$, $v' = 0$, $v = 0$, $w = 0$ and $\phi = 0$ for $z = 0$, $z = l$.

Following the solution of Volterra and Morell and of Den Hartog for partial rings, suitable expressions for the displacements satisfying the above boundary conditions are :

$$
\begin{aligned}
u = l \sin(\Omega t)\{\, & a_1 [1 - \cos(\frac{2\pi z}{l})] \\
& - a_2 [1 - \cos(\frac{4\pi z}{l})] \\
& - a_3 [1 - \cos(\frac{6\pi z}{l})]\}
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
v = l \sin(\Omega t)\{\, & b_1 [1 - \cos(\frac{2\pi z}{l})] \\
& - b_2 [1 - \cos(\frac{4\pi z}{l})] \\
& - b_3 [1 - \cos(\frac{6\pi z}{l})]\}
\end{aligned}
\tag{5.3}
$$

$$
w = l \sin(\Omega t) [c_1 \sin(\frac{\pi z}{l}) - c_2 \sin(\frac{2\pi z}{l}) - c_3 \sin(\frac{3\pi z}{l})]
\tag{5.4}
$$

$$
\phi = \sin(\Omega t) [d_1 \sin(\frac{\pi z}{l}) - d_2 \sin(\frac{2\pi z}{l}) - d_3 \sin(\frac{3\pi z}{l})]
\tag{5.5}
$$

where a_i, b_i, c_i and d_i are coefficients to be determined.

Upon substitution of these displacements into the strain energy function (5.1) and the kinetic energy function (3.4), an expression for the total energy for a rod undergoing free vibration is obtained :

$$
E.F. = U_{max} - T_{max}
\tag{5.6}
$$

This expression is not presented explicitly in this report because of its excessive length.

The total energy is then minimized with respect to the unknown coefficients by requiring

$$E.F._{,i} = 0 \tag{5.7}$$

where

$$i = a_1 , a_2 , a_3 , b_1 , b_2 , b_3 , c_1 , c_2 , c_3 , d_1 , d_2 , d_3$$

This yields a system of algebraic equations :

$$[\mathbf{M}_a - \Phi\, \mathbf{M}_b]\, \mathbf{C} = 0 \tag{5.8}$$

where

$$\mathbf{C}^T = [a_1 , a_2 , a_3 , b_1 , b_2 , b_3 , c_1 , c_2 , c_3 , d_1 , d_2 , d_3]$$

$$\Phi = \frac{\rho\, l^2\, \Omega^2}{E}$$

where Ω is the natural frequency of the free vibration of the rod.

$\mathbf{M}_a$, $\mathbf{M}_b$ and the integrated functions, co1 to co58, which appear in the matrix, are shown in Appendix B in FORTRAN codes. Since the coefficients of the column matrix C cannot be trivial, characteristic equations for the clamped–clamped case are obtained from the condition :

$$|\mathbf{M}_a - \Phi\, \mathbf{M}_b| = 0 \tag{5.9}$$

Next, the same process is repeated using normal modes of clamped–clamped

beams and bars as comparison functions.

As obtained from Young and Felger [20], the normal mode functions are :

$$u = l \sin(\Omega t) \{ a_1 \{ \cosh(\frac{k1\, z}{l}) - \cos(\frac{k1\, z}{l}) - kk1\, [\sinh(\frac{k1\, z}{l}) - \sin(\frac{k1\, z}{l})] \}$$
$$- a_2 \{ \cosh(\frac{k2\, z}{l}) - \cos(\frac{k2\, z}{l}) - kk2\, [\sinh(\frac{k2\, z}{l}) - \sin(\frac{k2\, z}{l})] \}\} \tag{5.10}$$

$$v = l \sin(\Omega t) \{ b_1 \{ \cosh(\frac{k1\, z}{l}) - \cos(\frac{k1\, z}{l}) - kk1\, [\sinh(\frac{k1\, z}{l}) - \sin(\frac{k1\, z}{l})] \}$$
$$- b_2 \{ \cosh(\frac{k2\, z}{l}) - \cos(\frac{k2\, z}{l}) - kk2\, [\sinh(\frac{k2\, z}{l}) - \sin(\frac{k2\, z}{l})] \}\} \tag{5.11}$$

$$w = l \sin(\Omega\, t)\, [c_1 \sin(\frac{\pi\, z}{l}) - c_2 \sin(\frac{2\,\pi\, z}{l})] \tag{5.12}$$

$$\phi = \sin(\Omega\, t)\, [d_1 \sin(\frac{\pi\, z}{l}) - d_2 \sin(\frac{2\,\pi\, z}{l})] \tag{5.13}$$

The coefficients of k1. kk1. k2 and kk2 [3], which appear in Equation (5.10) through (5.13) are given in Appendix C. Upon substituting Equation (5.10) through Equation (5.13) into the kinetic and strain energy functions. and repeating the minimization process. another set of characteristic equations is obtained.

$$\mathbf{M}_c - \Phi\, \mathbf{M}_d = 0 \tag{5.14}$$

where the elements of the M_c. M_d matrices and the coefficients are given in Appendix C.

Frequency results for an "almost straight" helical rod for the two sets of trial functions are compared with each other and with straight beam natural frequencies in Table 5-1.

		Helical Rod Eqs. 5-2, 5-3 5-4, and 5-5	Helical Rod Normal Mode Functions	Straight Beam
	u	0.2267×10^4	0.2263×10^4	0.2263×10^4
1st	v	0.2267×10^4	0.2263×10^4	0.2263×10^4
Mode	w	0.6355×10^6	0.6355×10^6	0.6355×10^6
	ϕ	0.4019×10^6	0.4019×10^6	0.4019×10^6
	u	0.1234×10^5	0.6238×10^4	0.6238×10^4
2nd	v	0.1234×10^5	0.6238×10^4	0.6238×10^4
Mode	w	0.1271×10^7	0.1271×10^7	0.1271×10^7
	ϕ	0.8038×10^6	0.8038×10^6	0.8038×10^6

Unit of Freq.= rad/sec, Length of The Rod = 1 inch
G/E = 0.4, LK = 1×10^{-8}, LL = 1×10^{-8}, LD = 1×10^3,
Density = 0.283 lb/inch^3

Table 5-1: Frequencies of Clamped–Clamped Straight-like Rod

For this structure, the first modes of the natural frequencies are almost identical for all three set of results. The frequencies corresponding to the second set show that the beam normal mode function approximation for the helical rod with small curvature gives frequencies more nearly equal to those of the straight beam and are therefore more accurate. Because the hyperbolic sine and cosine functions are mode functions for the straight beam, their use will give poor results for rods with large curvature. Thus, the first set of admissible functions would be expected to give better results for natural frequencies. That this is the case will be shown in Chapter 6.

5.3 Free Vibrations of a Clamped-Free Helical Rod with Constant Section Properties

In the clamped-free case, the boundary conditions are $u' = 0$, $u = 0$, $v' = 0$, $v = 0$, $w = 0$, $\phi = 0$ at $z = 0$ and

$$\frac{\partial F}{\partial u''} = 0$$

$$\frac{\partial}{\partial z}\frac{\partial F}{\partial u''} - \frac{\partial F}{\partial u'} = 0$$

$$\frac{\partial F}{\partial v''} = 0$$

$$\frac{\partial}{\partial z}\frac{\partial F}{\partial v''} - \frac{\partial F}{\partial v'} = 0$$

$$\frac{\partial F}{\partial w'} = 0$$

$$\frac{\partial F}{\partial \phi'} = 0$$

at $z = l$.

The relationships between the above expressions and the force and moment resultants acting on a cross section are not obvious. In Kingsbury's report [8, 9], the moments and forces are shown to be related to the displacement components in the following manner :

$$M_x = -E I_{xy} \alpha_y{}' - E I_{xx} (\alpha_z{}' - \kappa \phi) \tag{5.15}$$

$$M_y = E I_{yy} \alpha_y{}' - E I_{xy} (\alpha_z{}' - \kappa \phi) \tag{5.16}$$

$$M_z = G (I_{xx} - I_{yy}) \phi' - G I_{xx} \kappa \alpha_z + G I_{xy} \kappa \alpha_y \tag{5.17}$$

$$V_z = E\,A\,(\,w' - \kappa\,u) \tag{5.18}$$

$$V_x = -\,M_y{}' - \lambda\,M_z \tag{5.19}$$

$$V_y = M_x{}' - \lambda\,M_y - \kappa\,M_z \tag{5.20}$$

where

$$\alpha_x = u' - \lambda\,v - \kappa\,w$$

$$\alpha_y = -\,v' - \lambda\,u$$

By substituting α_x and α_y into Equation (5.15) through Equation (5.20), the following equations are obtained :

$$M_x = -\,E\,I_{xy}\,(\,u'' - \lambda\,v' - \kappa\,w'\,) - E\,I_{xx}\,(\,-v'' - \lambda\,u' - \kappa\,\phi\,) \tag{5.21}$$

$$M_y = E\,I_{yy}\,(\,u'' - \lambda\,v' - \kappa\,w'\,) - E\,I_{xy}\,(\,-v'' - \lambda\,u' - \kappa\,\phi\,) \tag{5.22}$$

$$M_z = G\,(\,I_{xx} - I_{yy})\,\phi' - G\,I_{xx}\,\kappa\,(\,v' - \lambda\,u\,) - G\,I_{xy}\,\kappa(u' - \lambda\,v - \kappa\,w\,) \tag{5.23}$$

$$V_z = E\,A\,(\,w' - \kappa\,u\,) \tag{5.24}$$

$$V_x = -\,E\,I_{yy}\,(\,u''' - \lambda\,v'' - \kappa\,w''\,) - E\,I_{xy}\,(\,-v''' - \lambda\,u'' - \kappa\,\phi'\,)$$

$$-\,\lambda\,E\,I_{xy}\,(\,u'' - \lambda\,v' - \kappa\,w'\,) - \lambda\,E\,I_{xx}\,(\,-v'' - \lambda\,u' - \kappa\,\phi\,) \tag{5.25}$$

$$V_y = -\,E\,I_{xy}\,(\,u''' - \lambda\,v'' + \kappa\,w''\,) + E\,I_{xx}\,(\,-v''' - \lambda\,u'' - \kappa\,\phi'\,)$$

$$-\,\lambda\,E\,I_{yy}\,(\,u'' - \lambda\,v' + \kappa\,w'\,) + \lambda\,E\,I_{xy}\,(\,-v'' - \lambda\,u' + \kappa\,\phi\,)$$

$$-\,G\kappa\,(\,I_{xx} - I_{yy})\,\phi' + G\,I_{xx}\kappa^2\,(\,v' + \lambda\,u\,) + G\,I_{xy}\kappa^2\,(\,u' - \lambda\,v + \kappa\,w\,) \tag{5.26}$$

Upon substitution of the strain energy function, Equation (5.1), boundary

conditions at the free end are found to be expressed in terms of moment and force resultants in the following manner:

$$\frac{\partial F}{\partial u''} = M_y = 0 \tag{5.27}$$

$$\frac{\partial F}{\partial v''} = -\, M_z = 0 \tag{5.28}$$

$$\frac{\partial F}{\partial w'} = V_z - \kappa\, M_y = 0 \tag{5.29}$$

$$\frac{\partial F}{\partial o'} = M_z = 0 \tag{5.30}$$

$$\frac{\partial}{\partial z}\frac{\partial F}{\partial u''} - \frac{\partial F}{\partial u'} = 0 \simeq -\, V_z \tag{5.31}$$

$$\frac{\partial}{\partial z}\frac{\partial F}{\partial v''} - \frac{\partial F}{\partial v'} = 0 \simeq -\, V_y \tag{5.32}$$

It is concluded that the forces and moments are zero or $u = u' = u'' = u''' = v = v' = v'' = v''' = w = w' = w'' = o = o' = 0$ at the free end ($z = l$). In this situation. it is very hard to find the suitable functions which fully satisfy the boundary conditions. Hence, the normal mode functions of a clamped-free straight beam[2] which are admissible trial functions are employed to get approximate solutions.

The normal mode functions for the straight clamped-free beam are :

$$u = l \sin(\Omega t)\{\, a_1 \{\cosh(\frac{k1\, z}{l}) - \cos(\frac{k1\, z}{l}) - kk1\,[\sinh(\frac{k1\, z}{l}) - \sin(\frac{k1\, z}{l})]\}$$
$$-\, a_2 \{\cosh(\frac{k2\, z}{l}) - \cos(\frac{k2\, z}{l}) - kk2\,[\sinh(\frac{k2\, z}{l}) - \sin(\frac{k2\, z}{l})]\}\} \tag{5.33}$$

---

[2] When the torsion λ and curvature κ become small, the helical rod will approach a straight beam and the boundary conditions will be much the same as those of the straight beam ($u'' = u''' = v'' = v''' = w' = o' = 0$).

$$v = l \sin(\Omega t) \{ b_1 \{ \cosh(\frac{k1\, z}{l}) - \cos(\frac{k1\, z}{l}) - kk1 [\sinh(\frac{k1\, z}{l}) - \sin(\frac{k1\, z}{l})] \}$$

$$- b_2 \{ \cosh(\frac{k2\, z}{l}) - \cos(\frac{k2\, z}{l}) - kk2 [\sinh(\frac{k2\, z}{l}) - \sin(\frac{k2\, z}{l})] \} \} \tag{5.34}$$

$$w = l \sin(\Omega\, t) [c_1 \sin(\frac{\pi\, z}{2\, l}) - c_2 \sin(\frac{3\, \pi\, z}{2\, l})] \tag{5.35}$$

$$o = \sin(\Omega\, t) [d_1 \sin(\frac{\pi\, z}{2\, l}) - d_2 \sin(\frac{2\, \pi\, z}{2\, l})] \tag{5.36}$$

By substituting the above expressions into the potential and kinetic energy functions. and repeating the steps as in the last section. the following equation is obtained:

$$M_e - \Phi M_f = 0 \tag{5.37}$$

where. the elements of the M_e. M_f matrices and the coefficients are shown in Appendix D.

The frequencies for clamped-free helical rods with circular cross sections. small curvature. and small torsion are tabulated in Table 5-2. These results are compared and found consistent with those of straight rods.

		Helical Rod	Straight Beam
		Normal Mode Functions	
	u	0.3556×10^3	0.3556×10^3
1st	v	0.3556×10^3	0.3556×10^3
Mode	w	0.3177×10^6	0.3177×10^6
	ϕ	0.2010×10^6	0.2010×10^6
	u	0.2229×10^6	0.2229×10^6
2nd	v	0.2229×10^6	0.2229×10^6
Mode	w	0.9532×10^6	0.9532×10^6
	ϕ	0.6029×10^6	0.6029×10^6

Unit of Freq.= rad/sec, Length of The Rod = 1 inch
G/E = 0.4, LK = $1\text{x}10^{-8}$, LL = $1\text{x}10^{-8}$, LD = $1\text{x}10^3$,
Density = 0.283 lb/inch^3

Table 5-2: Frequencies of Clamped-Free Straight-like Rod

CHAPTER 6

COMPARISON OF VARIOUS EQUATIONS AND THE STRAIN ENERGY FUNCTIONS FOR PARTIAL RING WITH CIRCULAR CROSS SECTION

Through comparison of the equations of motion of curved rods derived in this report with those derived earlier by Kingsbury [8, 9], slight differences are noted between the two. These differences do not represent errors in one or the other formulation but result instead from the way in which the equilibrium and kinematic terms are combined and the order in which the various simplifying assumptions are invoked. This chapter assesses the effect of these differences on the vibration of a circular ring which is a special case of general space curvature for which the torsion is zero and the curvature is constant.

The present equations are first compared with those of Kingsbury and with those presented by A. E. H. Love [10] for the circular ring. It is found that the most essential differences appear in the equations governing out-of-plane vibrations with those of Kingsbury and Love being most nearly in agreement.

The quantitative effect of these differences is assessed by employing the suitably simplified version of the strain energy functions derived in chapter 2 with a corresponding one presented by Love to obtain Rayleigh-Ritz solutions for the natural frequencies of out-of-plane vibrations of a clamped-clamped ring.

6.1 Comparison of Force Equilibrium Equation

6.1.1 Derivation of Governing Equations using Strain Energy Function

Upon applying the equations given in Section 4.5, eliminating all the terms containing I_{xy}, torsion λ, or their derivatives, and dropping the terms containing the derivatives of the area moment of inertias I_{xx}, I_{yy} and area A, the following equations are obtained :

$$EI_{yy}\kappa w''' - (AE\kappa - GI_{yy}\kappa^3)w' - EI_{yy}u'''' - GI_{yy}\kappa^2 u'' + AE\kappa^2 u = -A\rho\ddot{u} \tag{6.1}$$

$$-(EI_{yy}\kappa^2 - AE)w'' - GI_{yy}\kappa^4 w - EI_{yy}\kappa u''' - (GI_{yy}\kappa^3 + AE\kappa)u' = -A\rho\ddot{w} \tag{6.2}$$

$$EI_{xx}v'''' - GI_{xx}\kappa^2 v'' - (G-E)I_{xx}\kappa\phi'' = -A\rho\ddot{v} \tag{6.3}$$

$$-(G-E)I_{xx}\kappa v'' - (I_{yy}-I_{xx})G\phi'' - EI_{xx}\kappa^2\phi = -(I_{yy}-I_{xx})\rho\ddot{\phi} \tag{6.4}$$

It is noted that for the ring problem, the equations governing in-plane motions (Eq. (6.1) and Eq. (6.2)) uncouple from those of out-of-plane motions.

6.1.2 Governing Equations from Kingsbury's Formulation

$$EI_{yy}\kappa w''' - AE\kappa w' - EI_{yy}u'''' - AE\kappa^2 u = -A\rho\ddot{u} \tag{6.5}$$

$$-(EI_{yy}\kappa^2 - AE)w'' - EI_{yy}\kappa u''' - AE\kappa u' = -A\rho\ddot{w} \tag{6.6}$$

$$EI_{xx}v'''' - GI_{xx}\kappa^2 v'' - (G(I_{xx}+I_{yy}) + EI_{xx})\kappa\phi'' = -A\rho\ddot{v} \tag{6.7}$$

$$-(G-E)I_{xx}\kappa v'' - (I_{yy}-I_{xx})G\phi'' + EI_{xx}\kappa^2\phi = -(I_{yy}+I_{xx})\ddot{\phi}\rho \tag{6.8}$$

Through comparisons of Equations (6.1) and (6.2) with (6.5) and (6.6), differences between these two pairs are seen to involve terms of the order of the square or higher power of the curvature. These terms should not have a significant effect on the results for rods with small curvature. Equation (6.7) contains an extra term in the coefficient of ϕ'' which causes its magnitude to be approximately twice that of the corresponding coefficient of Equation (6.3). The practical effect of this difference can only be assessed numerically. Equations (6.3) and (6.4) lead to a symmetric stiffness matrix in the characteristic equation for the eigenvalues while Equations (6.7) and (6.8) do not.

6.1.3 Governing Equations based on Love's Assumptions

By using Love's equations of motion, stress resultants, and couples of the ring[3] [10], the out-of-plane vibration equations can be formulated as shown below :

$$EI_{xx}v'''' - G(I_{xx}-I_{yy})\kappa^2 v'' - (G(I_{xx}-I_{yy})-EI_{xx})\kappa\phi'' = -A\rho\ddot{v} \tag{6.9}$$

$$-(G-E)I_{xx}\kappa v'' - (I_{yy}-I_{xx})G\phi'' - EI_{xx}\kappa^2\phi = -(I_{yy}-I_{xx})\rho\ddot{\phi} \tag{6.10}$$

Equations (6.8) and (6.10) seem to be identical, while Equations (6.7) and (6.9) differ only by the difference between I_{xx} and I_{yy}. It may be concluded that Love's equations for out–of–plane vibration more clearly resemble those of Kingsbury.

In order to compare Equations (6.1) and (6.2) with the corresponding in–plane vibration equation derived by Love, it is necessary to simplify the former equations by imposing the condition of inextensibility of the ring axis. The vanishing of axial strain ϵ_{xx}

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[3] Equations 18, 19 and 20, p. 451.

implies :

$$\frac{\partial w}{\partial z} = \kappa \; u$$

Upon substituting the above relation into Equations (6.1) and (6.2), deleting the shear terms, then changing the independent variable from z to θ, which is the angle of the arc of the ring, the following equations are obtained :

$$E\,I_{yy}\,\kappa^4 \frac{\partial^6 w}{\partial \theta^6} - E\,I_{yy}\,\kappa^4 \frac{\partial^4 w}{\partial \theta^4} = -\,\rho\,A \frac{\partial^4 w}{\partial \theta^2\,\partial t^2} \qquad (6.11)$$

$$E\,I_{yy}\,\kappa^4 \frac{\partial^4 w}{\partial \theta^4} - E\,I_{yy}\,\kappa^4 \frac{\partial^2 w}{\partial \theta^2} = \rho\,A \frac{\partial^2 w}{\partial t^2} \qquad (6.12)$$

Adding Equations (6.11) and (6.12) yields the same equation as given in Love's book (page 452). If we integrate Equation (6.11) twice with respect to the variable θ, then add the result to Equation (6.12), we also find the constraints for stress resultants and the flexural couple vanish at the ends when the ring is incomplete[4] [10]. These results show that the equation found by Love can be obtained by simplifying and combining in-plane vibration Equations (6.1) and (6.2) through the relation of inextension of the ring axis.

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[4] p 452. "When the ring is incomplete the frequency equation is to be obtained by forming the conditions that N, T, G′ vanish at the ends".

6.2 Comparison of Strain Energy Functions

6.2.1 Out-of-Plane Vibration Strain Energy Function of Circular Ring

To obtain the strain energy for the circular ring from Equation (2.17), λ, I_{xy}, and the derivatives of κ are set equal to zero. This results in the following expression :

$$S.E.F._{CircularRing\ Out-of-Plane\ Vibration} =$$

$$\frac{E}{2} I_{xx} (v'')^2 - E I_{xx} \kappa\, o\, v'' - \frac{1}{2} G I_{xx} \kappa^2 (v')^2$$

$$- G I_{xx} \kappa\, o'\, v' - \frac{1}{2} E I_{xx} \kappa^2 o^2 - \frac{1}{2} G (I_{xx} - I_{yy}) (o')^2 \tag{6.13}$$

6.2.2 Out-of-Plane Vibration Strain Energy Function of Circular Ring Derived from Love's Formulation

According to Love [10], the total strain energy of the circular ring is :

$$U = \frac{1}{2} \left(\int_0^l \frac{\mathbf{G}^2}{E I_{xx}} - \frac{\mathbf{H}^2}{G (I_{xx} - I_{yy})} dz \right) \tag{6.14}$$

where **G** and **H** are the torsional and bending moment[5] [10] given by Equations (6.15) and (6.16).

$$\mathbf{G} = E I_{xx} (\kappa\, o - v'') \tag{6.15}$$

$$\mathbf{H} = G (I_{xx} - I_{yy}) (\kappa\, v' - o') \tag{6.16}$$

Substitution of Equations (6.15) and (6.16) into equation (6.14) yields the strain

[5] equation 20, p.451.

energy function (6.17).

$$S.E.F._{CircularRing\ Out-of-Plane\ Vibration} =$$

$$\frac{E}{2} I_{zz} (v'')^2 - E I_{zz} \kappa \phi v'' - \frac{1}{2} G (I_{zz} - I_{yy}) \kappa^2 (v')^2$$

$$- G (I_{zz} - I_{yy}) \kappa \phi' v' - \frac{1}{2} E I_{zz} \kappa^2 \phi^2 - \frac{1}{2} G (I_{zz} - I_{yy}) (\phi')^2 \tag{6.17}$$

Comparing the two strain energy functions. Equations (6.13) and (6.17), we conclude that the coefficients of the $(v')^2$ and $\phi' v'$ terms differ by a factor of two if $I_{zz} = I_{yy}$. The quantitative effect of these differences on the lowest frequency. f[6]. for the out-of-plane vibration of the clamped–clamped ring is tabulated in Table 6-1.

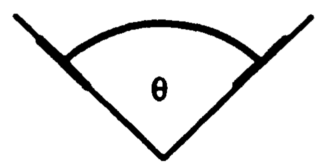
In Table 6-1. the first and second sets of data are the results of the substitution of admissible functions. given by Equation (5.2) through Equation (5.5). into the strain energy expressions (6.13) and (6.17) respectively. For the purpose of comparison. the results of the substitution of the normal mode functions (admissible functions) into Equation (6.17) are shown in the third set of data.

The three sets of data will be compared. In the first set of data. results for $\theta \geq 180^\circ$ are the same as the data presented by Den Hartog [6][7]. Comparing the first and second sets of data. it is found that the differences of the two shear strain energy terms will affect f to the maximum 15 %. Here, we may conclude that the two formulations can yield different results for large included angles, but there is no way of assessing their relative

[6] $f = (2/(DD\,\theta))\sqrt{(\Phi)}$

[7] Den Hartog's energy function is the same as equation (6.17)

	Equation (6-17)		Equation (6-13)		Equation (6-13)	
	Admissible functions Eqs. 5-2, 5-3, 5-4 and 5-5				Normal Mode Functions	
	DD=0.02	DD=0.1	DD=0.02	DD=0.1	DD=0.02	DD=0.1
θ	f	f	f	f	f	f
10	734.2	227.8	734.8	227.9	733.7	227.9
20	183.0	113.9	183.3	114.2	183.2	114.2
30	80.87	73.12	81.19	74.36	81.19	74.36
40	45.14	44.06	45.45	44.75	45.51	44.81
50	28.62	28.18	28.92	28.64	29.00	28.73
60	19.66	19.42	19.95	19.80	20.04	19.90
70	14.27	14.12	14.54	14.45	14.65	14.56
80	10.78	10.69	11.05	10.99	11.16	11.10
90	8.410	8.337	8.654	8.613		8.731
100	6.708	6.662	6.949	6.920		7.041
110	5.464	5.431	5.693	5.672		5.795
120	4.524	4.500	4.742	4.726		4.852
130	3.799	3.781	4.006	3.994		4.121
140	3.230	3.216	3.425	3.416		3.544
150	2.774	2.764	2.959	2.954		3.081
160	2.406	2.398	2.581	2.575		2.705
170	2.104	2.097	2.269	2.265		2.252
180	1.853	1.848	2.009	2.006		2.137
190	1.644	1.640	1.791	1.789		
200	1.467	1.463	1.606	1.604		
210	1.316	1.314	1.448	1.447		
220	1.187	1.185	1.312	1.311		
230	1.076	1.074	1.195	1.194		
240	.9795	.9780	1.092	1.091		
250	.8954	.8941	1.002	1.001		
260	.8216	.8206	.9231	.9224		
270	.7567	.7559	.8531	.8526		
280	.6994	.6987	.7910	.7905		
290	.6486	.6479	.7356	.7352		
300	.6034	.6028	.6861	.6857		
310	.5631	.5626	.6416	.6413		
320	.5270	.5266	.6016	.6013		
330	.4948	.4944	.5654	.5652		
340	.4659	.4655	.5327	.5325		
350	.4399	.4396	.5301	.5029		
360	.4166	.4163	.4761	.4760		



DD = Diameter of Cross Section / Diameter of Ring

Table 6-1: Frequencies of Out-of-Plane Vibration, Clamped Two Ends Partial Ring (G/E=0.4)

accuracy from the published literature. From the other viewpoint, it is found that when the angle of the ring is small, the error of the coefficient f between the first and second sets of data is small also. From the factor of two mentioned, it may be concluded that when the angle of the ring is small, the shear strain terms do not overly affect the accuracy of frequency.

From Table 6-1, we also found that most of the frequency parameter (f) values obtained by using normal mode functions of straight rods are slightly higher than those which were obtained from the other set of admissible functions. Missing data are negative eigenvalues which probably result from use of straight beam mode shapes and from the truncation errors of the IMSL[8] program.

[8] EIGZF subroutine, IMSL INC, Huston, Texas.

CHAPTER 7

VIBRATION OF THE CLAMPED-FREE HELICAL RODS

In this chapter, the Rayleigh–Ritz formulation of Chapter 5 for clamped–free helical rods is used to examine the effects of curvature, torsion, cross section shape and orientation of cross section principal axes on the fundamental natural frequencies and mode shapes of a helical rod of elliptical cross section. In all cases the rod material is assumed to have a Poisson's ratio 0.3 (GE=G E=0.4).

7.1 Effect of Curvature and Torsion on Mode Shapes and Natural Frequencies for Rod with Uniform Circular Cross Section

In this section, the effects of slenderness, curvature, and torsion on the lowest frequency and mode shape are examined on rods of constant circular cross section. With a fixed value of LD (l radius of cross section), LK ($l \times \kappa$) and LL ($l \times \lambda$) are varied. LD is then again fixed at a new value and the calculations repeated. The results of these calculations for the fundamental natural frequency of the fixed–free helix are presented in Figures 7-1 and 7-2.

As expected, the fundamental frequency increases with decreasing LD. It is seen that the natural frequency is insensitive to changes in the curvature and torsion parameters for small values of these parameters. This effect is explored in more detail in Figure 7-2 which plots frequency in a linear rather than logarithmic scale. It is seen that the

fundamental natural frequency becomes noticeably dependent upon curvature and torsion when each of these parameters reaches the value of 10^{-2}.

The effects of changes of curvature and torsion on mode shape are explored in Table 7-1. At small values of both torsion and curvature, transverse displacement dominates the mode shape. As the curvature is increased, for constant small torsion, the displacements u and w become coupled as do v and ϕ. On the other hand, as the torsion parameter is increased with small curvature (moving down the left hand column) the lateral displacements u and v become coupled and approach the same magnitude. If both parameters are increased. then all of the variables approach the same magnitude. Significant coupling among the displacement variables seems to begin to occur at about the same values of the curvature and torsion parameters when frequency changes become noticeable.

7.2 Effect of Non-Circular Cross Section on Mode Shapes and Frequencies

In order to examine the effect of out–of–roundness on curved rod motion, the cross section of the rod is changed from circular to elliptical while keeping the cross section area constant. The configuration is shown in Figure 7-3. The product of inertia (I_{xy}) is then systematically varied by rotating the principal axes of the cross configurations while the torsion (LL) and curvature (LK) parameters are varied for each value of orientation of the principal axes of the cross section.

Figures 7-5 and 7-6 show the effect on fundamental natural frequency of changing the ratio of major to minor axes while keeping $\beta = 0$ (I_{xy}=0). Figure 7-4 shows the results

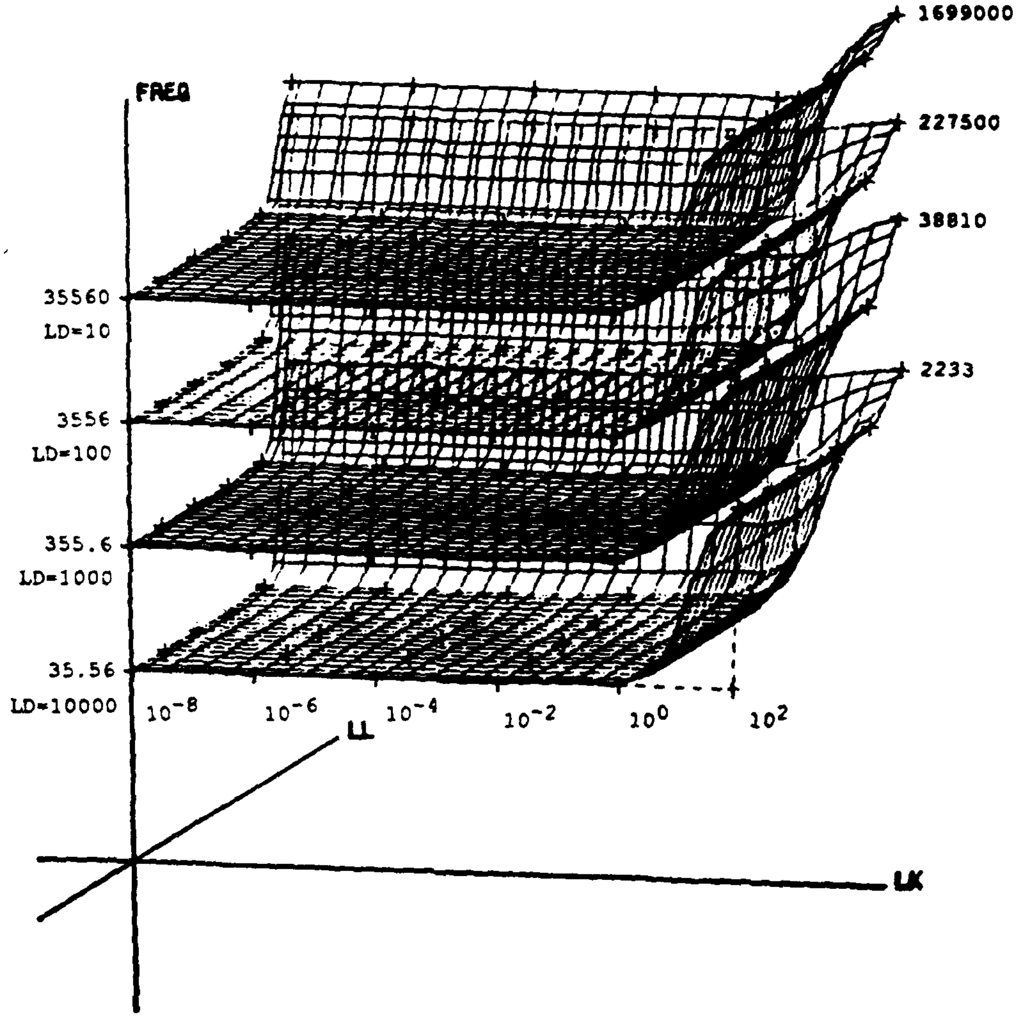


Figure 7-1: Frequencies of Clamped-Free Helical Rod.
(LD=1×10^1 to LD=1×10^4)

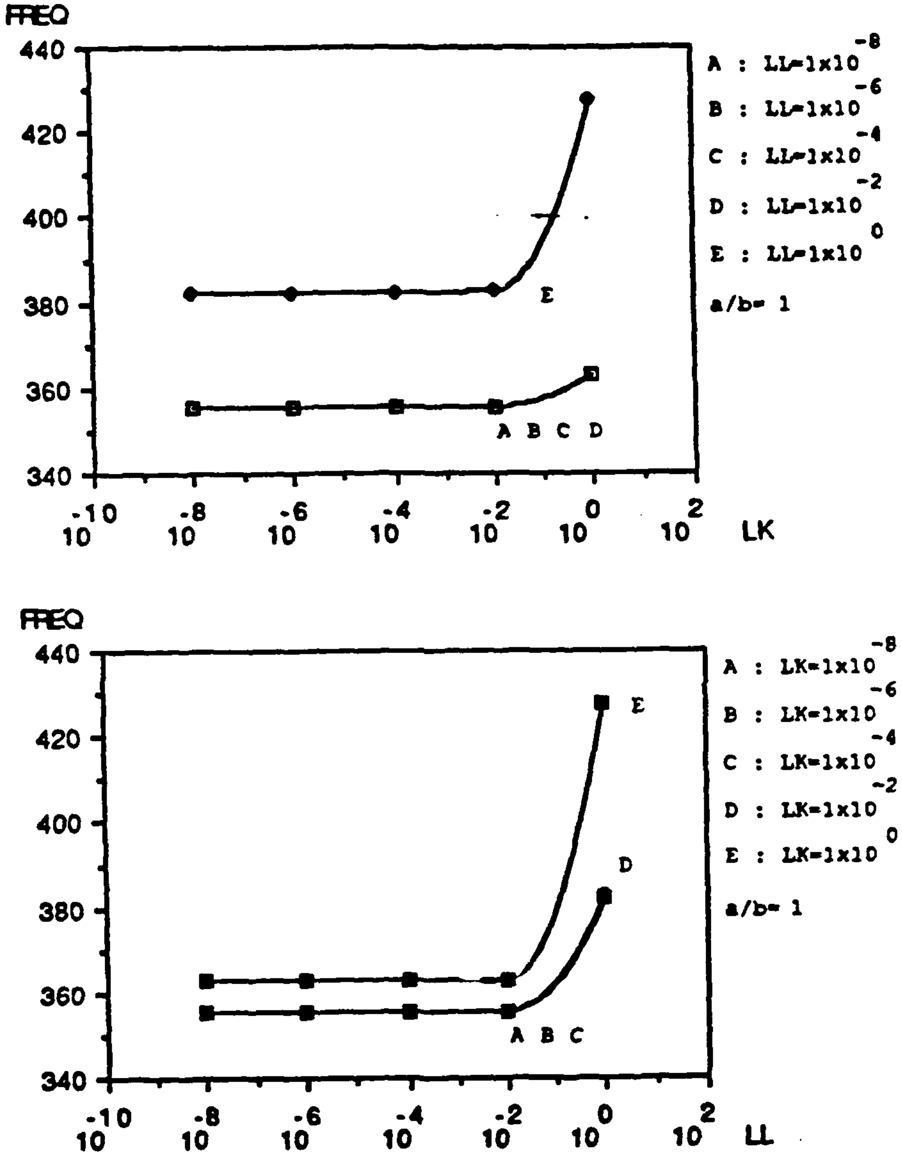


Figure 7-2: Frequencies of Clamped-Free Helical Rod.
($LD=1\times 10^3$)

LL 0.10E-07

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	355.6	355.6	355.6	355.6	363.1	10730
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.32E+08	-0.11E+10	-0.12E+10	-0.12E+10	-0.15E+12	0.82E+12
W	0.43E-08	0.89E-06	0.91E-04	0.91E-02	0.91E+00	0.11E+03
O	-0.13E+00	-0.47E+03	-0.48E+05	-0.49E+07	-0.51E+11	-0.35E+11

LL 0.10E-05

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	355.6	355.6	355.6	355.6	363.1	10730
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.13E+08	-0.12E+08	-0.12E+08	-0.12E+08	-0.15E+10	0.82E+10
W	0.99E-08	0.90E-06	0.91E-04	0.91E-02	0.91E+00	0.11E+03
O	-0.54E-01	-0.48E+01	-0.48E+03	-0.49E+05	-0.51E+09	-0.35E+09

LL 0.10E-03

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	355.6	355.6	355.6	355.6	363.1	10730
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.11E+06	-0.12E+06	-0.12E+06	-0.12E+06	-0.15E+08	0.82E+08
W	0.87E-08	0.91E-06	0.91E-04	0.91E-02	0.91E+00	0.11E+03
O	-0.44E-03	-0.48E-01	-0.48E+01	-0.49E+03	-0.51E+07	-0.35E+07

LL 0.10E-01

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	355.6	355.6	355.6	355.6	363.1	10730
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.13E+04	-0.12E+04	-0.12E+04	-0.12E+04	-0.15E+06	0.82E+06
W	0.98E-08	0.91E-06	0.91E-04	0.91E-02	0.91E+00	0.11E+03
O	-0.55E-05	-0.48E-03	-0.48E+01	-0.49E+01	-0.51E+05	-0.35E+05

LL 0.10E+01

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	382.2	382.2	382.2	382.7	427.7	10790
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.12E+02	-0.12E+02	-0.12E+02	-0.12E+02	-0.16E+04	0.81E+04
W	0.91E-08	0.91E-06	0.91E-04	0.91E-02	0.91E+00	0.11E+03
O	-0.59E-07	-0.59E-05	-0.59E-03	-0.60E-01	-0.50E+03	-0.34E+03

LL 0.10E+03

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	16950	16950	16950	16950	17550	38810
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.50E+02	-0.53E+02	-0.53E+02	-0.53E+02	-0.69E+02	0.69E+02
W	0.89E-08	0.91E-06	0.91E-04	0.91E-02	0.91E+00	0.94E+02
O	-0.36E-06	-0.39E-04	-0.39E-02	-0.39E+00	-0.50E+02	0.11E+02

Table 7-1: Fundamental Frequencies and Mode Shapes of Clamped-Free Helical Rod. (LD=1×10^{3}, a/b=1)

for a circular cross section to provide a basis for comparison.

It is seen that the effect of out-of-roundness is not symmetrical because the direction of the major axis substantially affects the manner in which the fundamental natural frequency is affected by changes of the torsion and curvature parameters.

Displacement coupling caused by the non-axisymmetric cross section is explored in Tables 7-2 through 7-4 and Figures 7-5. 7-6. and 7-7. Figures 7-5, 7-6. and 7-7 illustrate the effects of cross section rotation on fundamental natural frequency. Again, it is noted that rotating the cross section of fixed geometry changes the manner in which the curvature and torsion parameters affect the frequency.

Comparing Tables 7-2 through 7-4. it is first noted that for the almost straight rod ($LL=LK=10^{-8}$) the dominant displacement remains in the direction of the minor axis as the cross section is rotated. For higher values of curvature and torsion. rotation of the cross section is seen to change the relative values and phase of the four displacement components.

Finally, it is noted that when $LD=1000$, LK and LL are smaller than 10^{-4} and a/b is in the range of 1.1 to 1. then the fundamental frequencies are independent of the rotation angle β and also lower than the frequencies of a rod with circular cross section.

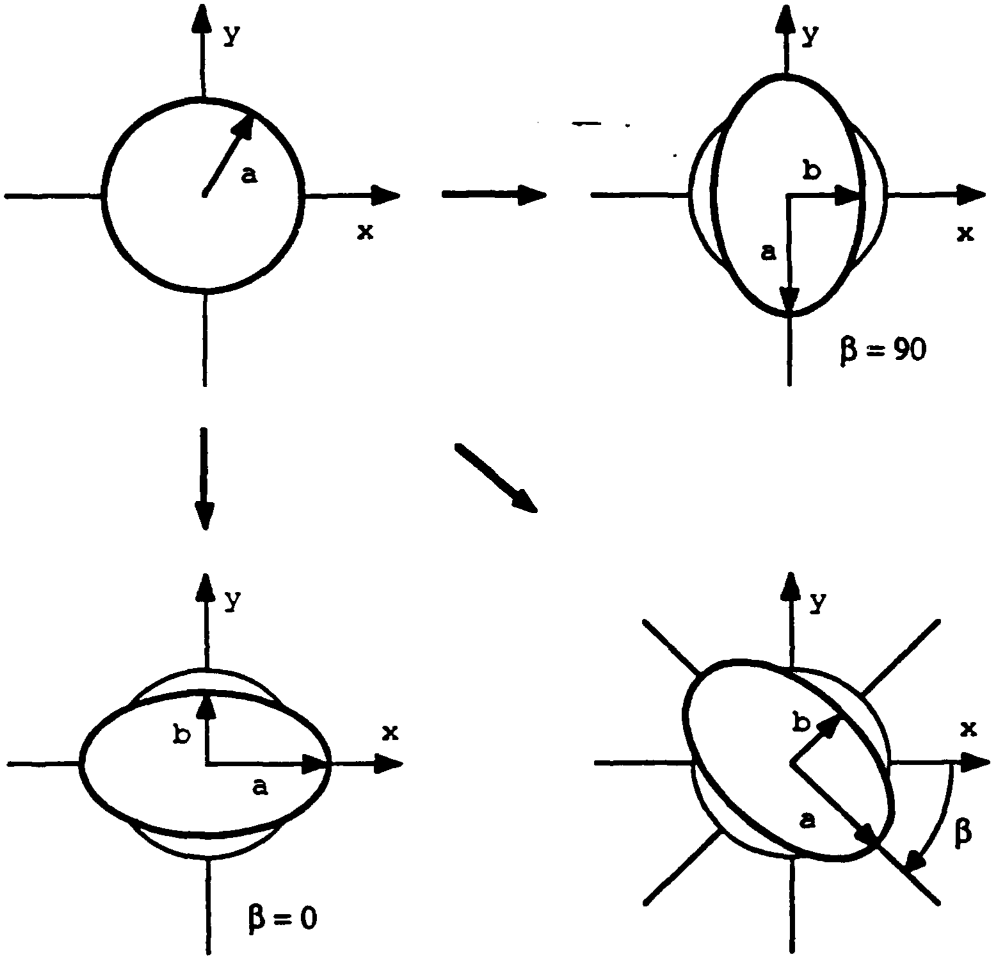


Figure 7-3: Deformation of Cross Section

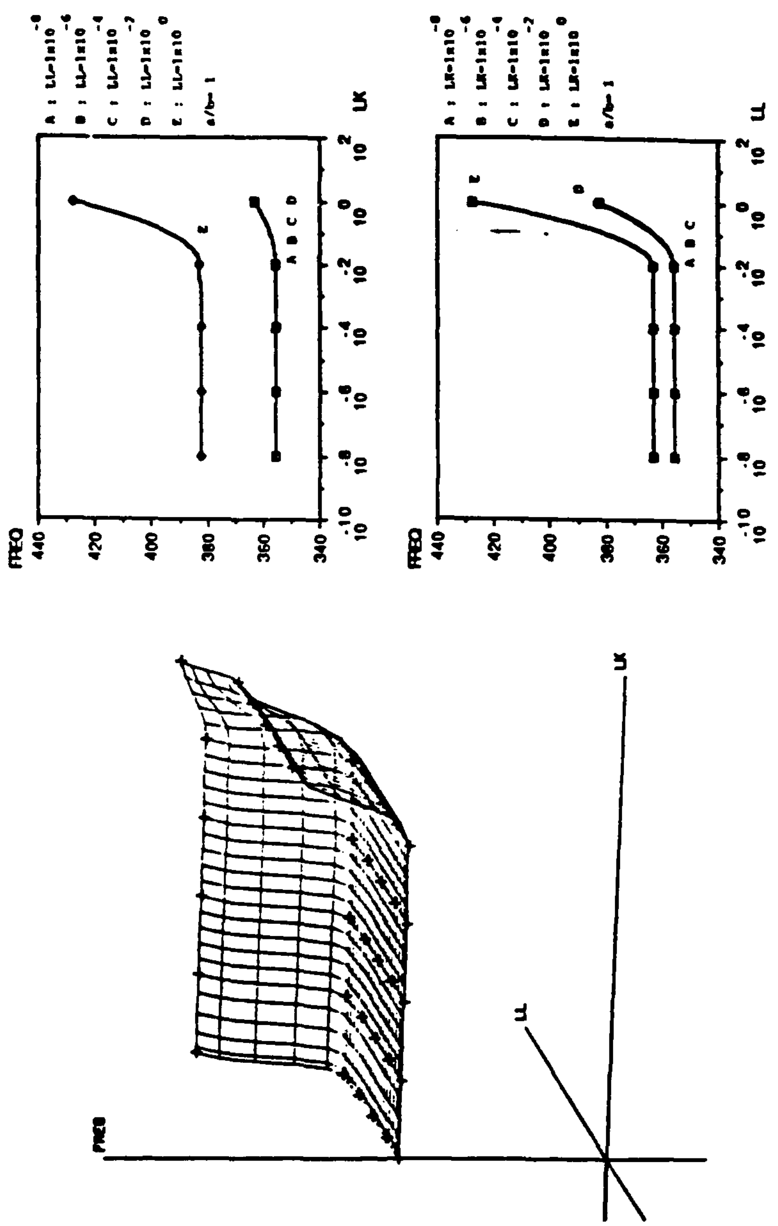


Figure 7-4: Effects of Non-Circular Cross Section On the Fundamental Frequency (a/b=1)

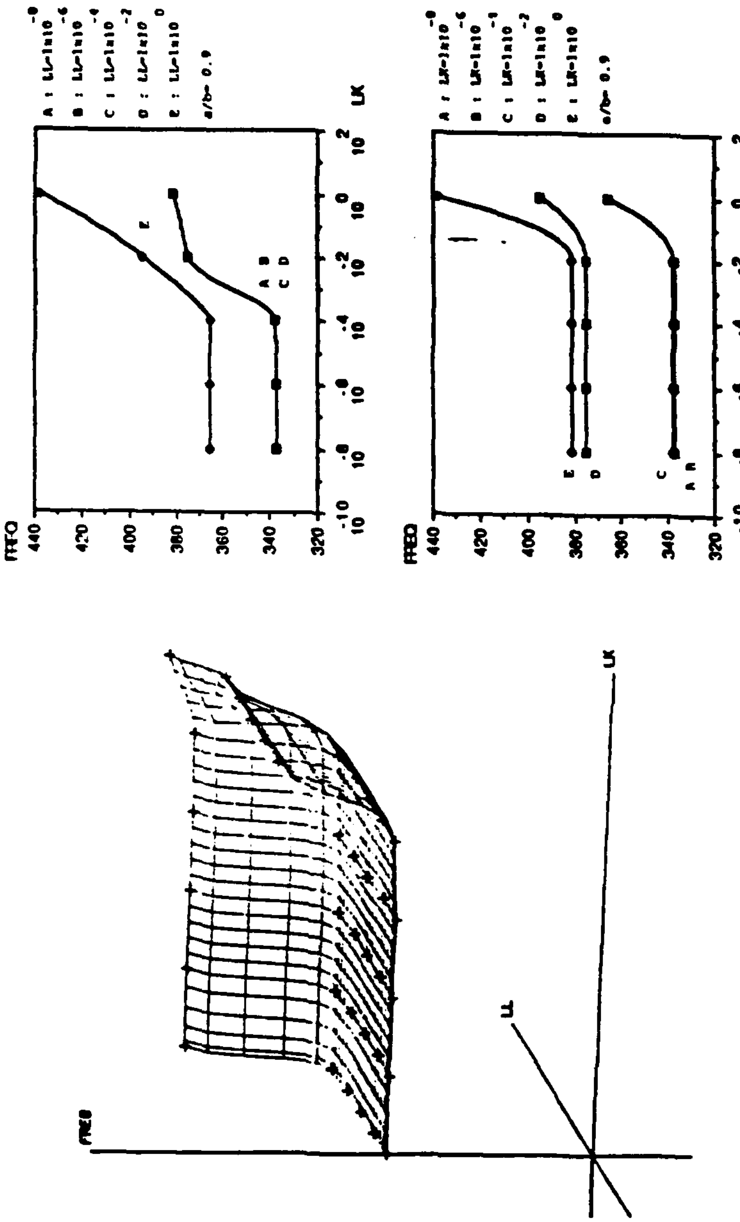


Figure 7-5: Effects of Non–Circular Cross Section On the Fundamental Frequency (a/b=0.9, β=0° or a/b=1.1, β=90°)

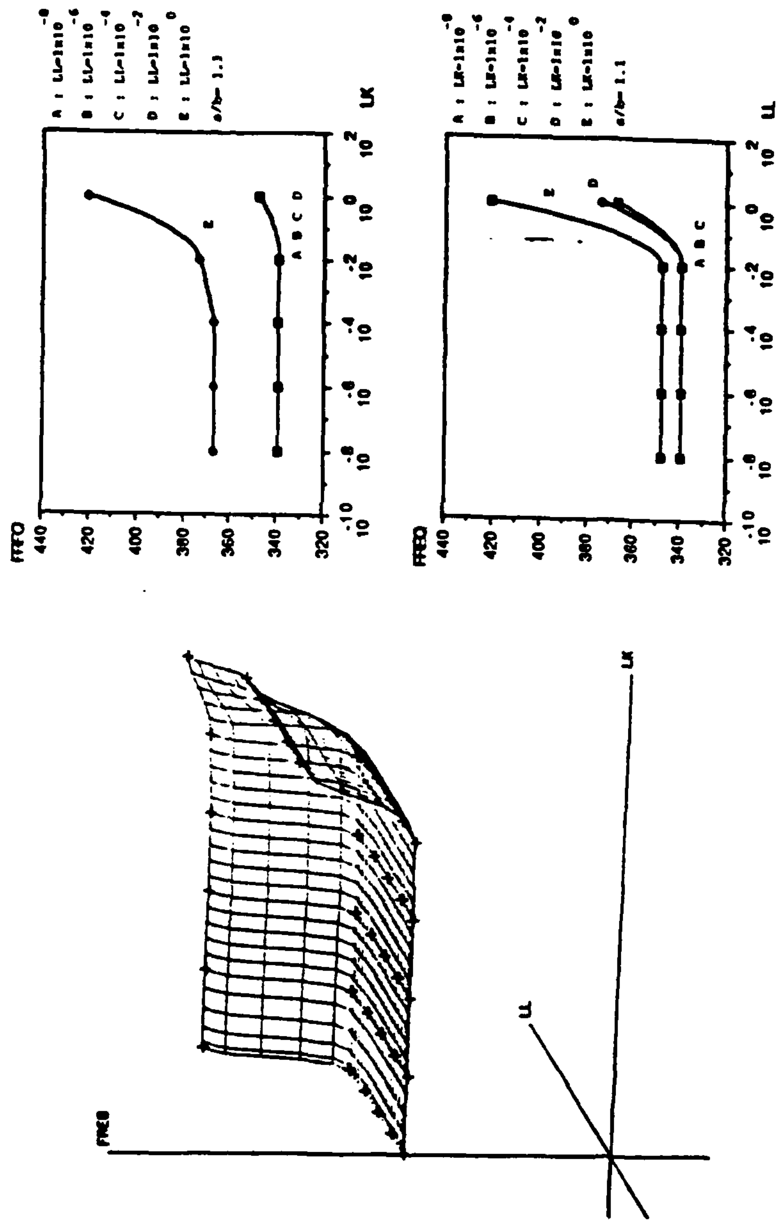


Figure 7-6: Effects of Non-Circular Cross Section On the Fundamental Frequency (a/b=1.1, β=0°)

LL 0.10E-07

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.1	339.1	347.9	10230
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.33E+09	0.33E+09	0.33E+09	-0.13E+10	-0.15E+12	0.83E+12
W	0.87E-08	0.26E-06	0.25E-04	0.94E-02	0.91E+00	0.12E+03
O	0.12E+01	0.12E+03	0.12E+05	-0.50E+07	-0.44E+11	-0.35E+11

LL 0.10E-05

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.1	339.1	347.9	10230
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.33E+07	0.33E+07	0.33E+07	-0.13E+08	-0.15E+10	0.83E+10
W	0.24E-08	0.25E-06	0.25E-04	0.94E-02	0.91E+00	0.12E+03
O	0.12E-01	0.12E+01	0.12E+03	-0.50E+05	-0.44E+09	-0.35E+09

LL 0.10E-03

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.1	339.1	347.9	10230
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.33E+05	0.33E+05	0.33E+05	-0.13E+06	-0.15E+08	0.83E+08
W	0.25E-08	0.25E-06	0.25E-04	0.94E-02	0.91E+00	0.12E+03
O	0.12E-03	0.12E-01	0.12E+01	-0.50E+03	-0.44E+07	-0.35E+07

LL 0.10E-01

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.1	339.1	347.9	10230
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.33E+03	0.33E+03	0.33E+03	-0.13E+04	-0.15E+06	0.83E+06
W	0.25E-08	0.25E-06	0.25E-04	0.94E-02	0.91E+00	0.12E+03
O	0.12E-05	0.12E-03	0.12E-01	-0.50E+01	-0.44E+05	-0.35E+05

LL 0.10E+01

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	366.9	366.9	366.9	373.3	420.3	10300
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.26E+01	0.26E+01	0.26E+01	-0.14E+02	-0.15E+04	0.82E+04
W	0.29E-08	0.29E-06	0.29E-04	0.95E-02	0.91E+00	0.12E+03
O	0.11E-07	0.11E-05	0.11E-03	-0.64E-01	-0.46E+03	-0.34E+03

LL 0.10E+03

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	16160	16160	16160	16190	18340	39170
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.23E-02	-0.23E-02	-0.23E-02	-0.26E-02	-0.58E+02	0.69E+02
W	0.54E-08	0.54E-06	0.54E-04	0.54E-02	0.91E+00	0.94E+02
O	0.43E-10	0.43E-08	0.43E-06	0.40E-04	-0.46E+02	-0.13E+02

Table 7-2: Effects of Non-Circular Cross Section On the Fundamental Frequency and Mode Shape ($a/b=1.1$, $\beta=0^{\circ}$)

LL 0.10E-07

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.2	356.2	364.0	10730
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.10E+01	0.10E+01	0.10E+01	0.94E+02	0.31E+05	-0.17E+06
W	0.39E-08	0.39E-06	0.39E-04	0.54E-02	0.90E+00	-0.23E+03
O	0.38E-08	0.38E-06	0.38E-04	0.39E+00	0.10E+05	0.74E+04

LL 0.10E-05

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.2	356.2	364.0	10730
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.10E+01	0.10E+01	0.10E+01	0.94E+02	0.31E+05	-0.17E+06
W	0.39E-08	0.39E-06	0.39E-04	0.54E-02	0.90E+00	-0.23E+03
O	0.38E-08	0.38E-06	0.38E-04	0.39E+00	0.10E+05	0.74E+04

LL 0.10E-03

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.2	356.2	364.0	10730
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.10E+01	0.10E+01	0.10E+01	0.95E+02	0.31E+05	-0.17E+06
W	0.39E-08	0.39E-06	0.39E-04	0.54E-02	0.90E+00	-0.23E+03
O	0.38E-08	0.38E-06	0.38E-04	0.39E+00	0.10E+05	0.74E+04

LL 0.10E-01

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.2	356.4	364.1	10730
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.99E+00	0.99E+00	0.10E+01	0.10E+03	0.39E+05	-0.22E+06
W	0.39E-08	0.39E-06	0.39E-04	0.51E-02	0.89E+00	-0.32E+03
O	0.38E-08	0.38E-06	0.38E-04	0.43E+00	0.13E+05	0.94E+04

LL 0.10E+01

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	366.9	366.9	367.1	395.6	436.2	10850
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.44E+00	0.44E+00	0.45E+00	-0.14E+02	-0.17E+04	0.85E+04
W	0.38E-08	0.38E-06	0.38E-04	0.94E-02	0.91E+00	0.13E+03
O	0.19E-08	0.19E-06	0.19E-04	-0.78E-01	-0.67E+03	-0.35E+03

LL 0.10E+03

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	16160	16160	16160	16170	17540	38960
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.10E+01	-0.10E+01	-0.10E+01	-0.10E+01	-0.25E+02	0.69E+02
W	0.54E-08	0.54E-06	0.54E-04	0.54E-02	0.88E+00	0.94E+02
O	-0.65E-08	-0.65E-06	-0.65E-04	-0.66E-02	-0.18E+02	0.12E+02

Table 7-8: Effects of Non-Circular Cross Section On the Fundamental Frequency and Mode Shape ($a/b=1.1$, $\beta=45°$)

LL 0.10E-07

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.3	373.0	379.0	11260
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.47E-07	-0.47E-07	-0.47E-07	-0.51E+09	-0.11E+12	0.59E+12
W	0.39E-08	0.39E-06	0.39E-04	0.71E-02	0.90E+00	0.96E+01
O	0.56E-16	-0.17E-13	-0.17E-11	-0.23E+07	-0.40E+11	-0.25E+11

LL 0.10E-05

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.3	373.0	379.0	11260
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.35E-06	-0.35E-06	-0.35E-06	-0.10E+08	-0.16E+10	0.80E+10
W	0.39E-08	0.39E-06	0.39E-04	0.87E-02	0.91E+00	0.11E+03
O	-0.11E-14	-0.90E-13	-0.92E-11	-0.47E+05	-0.57E+09	-0.34E+09

LL 0.10E-03

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.3	373.0	379.0	11260
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.31E-04	-0.31E-04	-0.31E-04	-0.10E+06	-0.16E+08	0.81E+08
W	0.39E-08	0.39E-06	0.39E-04	0.87E-02	0.91E+00	0.11E+03
O	-0.74E-13	-0.74E-11	-0.75E-09	-0.47E+03	-0.58E+07	-0.34E+07

LL 0.10E-01

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	339.1	339.1	339.3	373.0	379.0	11260
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.31E-02	-0.31E-02	-0.31E-02	-0.10E+04	-0.16E+06	0.81E+06
W	0.39E-08	0.39E-06	0.39E-04	0.87E-02	0.91E+00	0.11E+03
O	-0.74E-11	-0.74E-09	-0.75E-07	-0.47E+01	-0.58E+05	-0.34E+05

LL 0.10E+01

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	366.9	366.9	367.1	393.1	436.6	11310
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	-0.38E+00	-0.38E+00	-0.39E+00	-0.11E+02	-0.16E+04	0.81E+04
W	0.42E-08	0.42E-06	0.42E-04	0.87E-02	0.91E+00	0.11E+03
O	-0.14E-08	-0.14E-06	-0.14E-04	-0.56E-01	-0.54E+03	-0.34E+03

LL 0.10E+03

LK	0.10E-07	0.10E-05	0.10E-03	0.10E-01	0.10E+01	0.10E+03
FREQ	16160	16160	16160	16160	16790	38530
U	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01	0.10E+01
V	0.44E+03	0.44E+03	-0.27E-01	0.50E+03	-0.79E+02	0.69E+02
W	-0.23E-07	-0.23E-05	-0.20E-03	-0.26E-01	0.92E+00	0.94E+02
O	0.29E-05	0.29E-03	0.23E-04	0.33E+01	-0.52E+02	0.92E+01

Table 7-4: Effects Non–Circular Cross Section On the Fundamental Frequency and Mode Shape (a/b=1.1, β=90°)

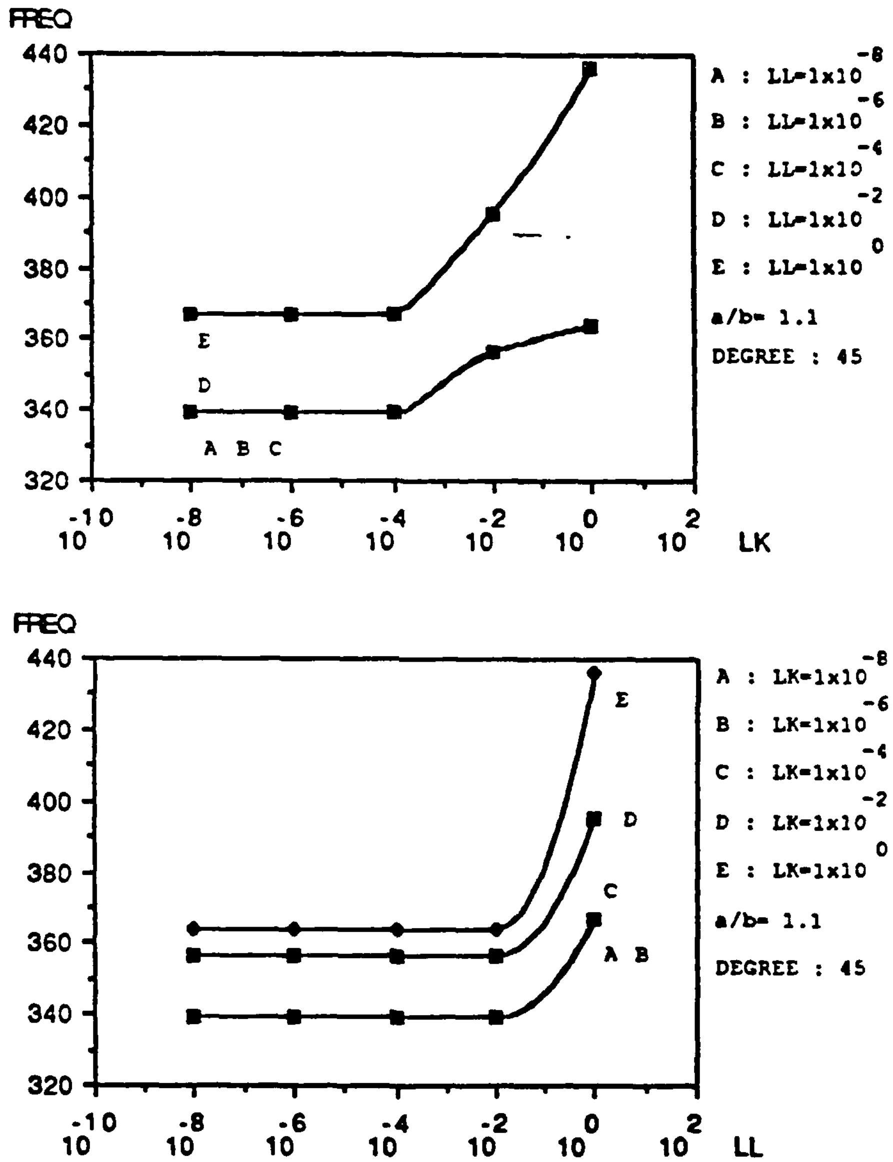


Figure 7-7: Effects of Non–Circular Cross Section On the Fundamental Frequency ($a/b=1.1, \beta=45^{\circ}$)

CHAPTER 8

CONCLUSIONS

The primary purpose of the work presented in this report was to examine the effects of curvature in three dimensional space and of out-of-roundness on the vibrations of rods with special attention to rods which are nearly straight.

Although the derived governing equations apply to a rod with general space curvature. the mode shapes and natural frequencies of a free vibration curve were calculated for a rod whose centroidal axis was a helix. For this case. both the curvature and torsion were constants rather than position dependent. Natural frequencies and mode shapes were obtained using three term series with the Raleigh-Ritz method.

It was found that the introduction of even slight space curvature, as characterized by the curvature and torsion parameters. introduced significant changes on rod natural frequencies and especially on mode shapes compared to those of a straight rod. Increasing either curvature parameter always increased fundamental natural frequency and degree of coupling among the displacement variables in the natural modes.

The effect of out-of-roundness was examined by calculating the mode shapes and natural frequencies of a rod of ellipitical cross sections and varying the orientation of the principal axes of the ellipse with respect to the natural coordinates of space curvature of the

rod center line. It was found that the orientation of the cross section had a pronunced effect on the coupling of the displacement variables on any given natural mode of free vibration. and that out-of-roundness changed the manner in which changes of the torsion and curvature parameters affected the rod's natural frequencies and mode shapes.

The mode shape and natural frequencies were calculated by the Raleigh-Ritz methods which, in turn, required use of the strain and kinetic energy functions for a helical rod. These were obtained as a special case for the strain energy and kinetic energy of a rod with general space curvature and an arbitrary cross section shape which was first derived. The general strain and kinetic energy expressions were also used to formulate the governing equations and natural boundary conditions of a rod with general space curvature. These equations were found to differ in certain respects from similiar equations obtained earlier by Kingsbury. It was found that the latter equations did not yield a symmetric stiffness matrix when a natural mode solution was introduced but the newly derived ones did.

The derived equations were also compared to those obtained by Love [10] for a circular ring. Imposing the condition of inextensibility of the centroidal axis for a plane ring yielded the same in-plane vibration equation as given by Love. In the out-of-plane vibration equation set, Love's equations of motion were identical with Kingsbury's, but slightly different from those derived in this report. As a further check on the derived energy expression, a Raleigh-Ritz solution was formulated for a partial circular ring and compared with results presented by Den Hartog who employed Love's energy function. It was found that the energy expression derived in this report yields frequencies slightly higher than Den Hartog's, especially as the arc angle of the ring became larger. There is no way, however, of assessing the relative accuracy of the different energy expressions from the

published literature.

APPENDIX A

STRAIN ENERGY FUNCTION

P_1, P_2, P_3 and P_4 appearing in equation (2.13) are shown below.

$$P_1 = E\kappa v''w' - E\kappa\lambda u'w' - E\kappa\lambda' uw' - E\kappa^2\phi w' - E\kappa' v''w$$

$$- G\kappa^3 v'w - E\kappa'\lambda u'w - E\kappa'\lambda' uw - G\kappa^3\lambda uw - G\kappa^2\phi' w$$

$$- E\kappa\kappa'\phi w - E\lambda v'v'' - E\lambda' vv'' - Eu''v'' - G\kappa^2\lambda vv'$$

$$- E\lambda^2 u'v' - G\kappa^2 u'v' - E\lambda\lambda' uv' - G\kappa\lambda\phi v' - E\kappa\lambda\phi v'$$

$$- E\lambda\lambda' u'v - E\lambda'^2 uv - G\kappa^2\lambda^2 uv - G\kappa\lambda\phi' v - E\kappa\lambda'\phi v$$

$$- E\lambda u'u'' - E\lambda' uu'' - E\kappa\phi u'' - G\kappa^2\lambda uu' - G\kappa\phi' u'$$

$$- G\kappa\lambda^2\phi u$$

$$P_2 = \frac{1}{2}\, Ev''^2 - G\kappa^2 v'^2 - E\lambda^2 u'^2 - E\lambda'^2 u^2 - G\kappa^2\lambda^2 u^2$$

$$- G\phi'^2 - G\lambda^2\phi^2 - E\kappa^2\phi^2$$

$$- E\lambda u'v'' - E\lambda' uv'' - E\kappa\phi v'' - G\kappa^2\lambda uv' - G\kappa\phi' v'$$

$$- E\lambda\lambda' uu' - E\kappa\lambda\phi u' - G\kappa\lambda\phi' u - E\kappa\lambda'\phi u$$

$$
\begin{aligned}
P_3 = \frac{1}{2}\, & E\kappa^2 w'^2 - E\kappa'^2 w^2 - G\kappa^4 w^2 - E\lambda^2 v'^2 - E\lambda'^2 v^2 \\
& - G\kappa^2 \lambda^2 v^2 - E u''^2 - G\kappa^2 u'^2 - G o'^2 - G\lambda^2 o^2 \\
& - E\kappa\kappa' w w' - E\kappa\lambda v' w' - E\kappa\lambda' v w' - E\kappa u'' w' - E\kappa' \lambda v' w \\
& - E\kappa' \lambda' v w - G\kappa^3 \lambda v w - E\kappa' u'' w + G\kappa^3 u' w + G\kappa^2 \lambda \phi w \\
& - E\lambda\lambda' v v' - E\lambda u'' v' - E\lambda' u'' v - G\kappa^2 \lambda u' v - G\kappa\lambda^2 \phi v \\
& - G\kappa\lambda o u'
\end{aligned}
$$

$$
P_4 = \frac{1}{2}\, E w'^2 - E\kappa^2 u^2 - E\kappa u w'
$$

where

$$
(\ )' = \frac{\partial (\ )}{\partial z}.
$$

APPENDIX B

MATRICES MA AND MB

In Appendices B, C, and D, the following dimensionless parameters have been introduced.

$$\text{pi} = \pi$$

$$\text{ge} = \frac{G}{E}$$

$$\text{lk} = l\,\kappa$$

$$\text{ll} = l\,\lambda$$

$$\text{la} = \frac{A\,l^2}{I_{xx}}$$

$$\text{xy} = \frac{I_{yy}}{I_{xx}}$$

$$\text{xz} = \frac{I_{zz}}{I_{xx}}$$

B.1 Matrix Ma

$$\mathbf{M}_a(i,j) = \mathbf{M}_a(j,i) \quad for \quad i \neq j \quad (i,j = 1,....,12)$$

```
ma(1,1) = (16*co28*pi**4+4*co20*ge*lk**2*pi**2)*xy+4*co20*ll**2*pi
1    **2+(co28+1)*ge*lk**2*ll**2+(co28+1)*la*lk**2

ma(1,2) = ge*lk**2*ll**2+la*lk**2

ma(1,3) = ge*lk**2*ll**2+la*lk**2

ma(1,4) = (16*co28*pi**4+(4*co20*ge*lk**2-4*co20*ll**2)*pi**2+(-co
1    28-1)*ge*lk**2*ll**2)*xz

ma(1,5) = -ge*lk**2*ll**2*xz

ma(1,6) = -ge*lk**2*ll**2*xz

ma(1,7) = (2*co12*lk*ll*pi**2-co4*ge*lk**3*ll)*xz

ma(1,8) = (8*co28*lk*pi**3+2*co20*ge*lk**3*pi)*xy+2*co28*la*lk*pi

ma(1,9) = (6*co23*lk*ll*pi**2-co29*ge*lk**3*ll)*xz

ma(1,10) = ((2*co12*ge-4*co4)*lk*pi**2-co4*ge*lk*ll**2)*xz

ma(1,11) = 2*co20*ge*lk*ll*pi*xy+(-2*co28*ge-2*co20)*lk*ll*pi

ma(1,12) = ((6*co23*ge-4*co29)*lk*pi**2-co29*ge*lk*ll**2)*xz

ma(2,2) = (256*co50*pi**4+16*co46*ge*lk**2*pi**2)*xy+16*co46*ll**2
1    *pi**2+(co50+1)*ge*lk**2*ll**2+(co50+1)*la*lk**2

ma(2,3) = ge*lk**2*ll**2+la*lk**2

ma(2,4) = -ge*lk**2*ll**2*xz

ma(2,5) = (256*co50*pi**4+(16*co46*ge*lk**2-16*co46*ll**2)*pi**2+(
1    -co50-1)*ge*lk**2*ll**2)*xz

ma(2,6) = -ge*lk**2*ll**2*xz

ma(2,7) = (4*co16*lk*ll*pi**2-co8*ge*lk**3*ll)*xz
```

```
ma(2,8) = 0

ma(2,9) = (12*co42*lk*ll*pi**2-co38*ge*lk**3*ll)*xz

ma(2,10) = ((4*co16*ge-16*co8)*lk*pi**2-co8*ge*lk*ll**2)*xz

ma(2,11) = 0

ma(2,12) = ((12*co42*ge-16*co38)*lk*pi**2-co38*ge*lk*ll**2)*xz

ma(3,3) = (1296*co55*pi**4+36*co53*ge*lk**2*pi**2)*xy+36*co53*ll**
1    2*pi**2+(co55+1)*ge*lk**2*ll**2+(co55+1)*la*lk**2

ma(3,4) = -ge*lk**2*ll**2*xz

ma(3,5) = -ge*lk**2*ll**2*xz

ma(3,6) = (1296*co55*pi**4+(36*co53*ge*lk**2 [illegible]o53*ll**2)*pi**2+
1    (-co55-1)*ge*lk**2*ll**2)*xz

ma(3,7) = (6*co18*lk*ll*pi**2-co10*ge*lk**3*ll)*xz

ma(3,8) = 0

ma(3,9) = (18*co44*lk*ll*pi**2-co40*ge*lk**3*ll)*xz

ma(3,10) = ((6*co18*ge-36*co10)*lk*pi**2-co10*ge*lk*ll**2)*xz

ma(3,11) = 0

ma(3,12) = ((18*co44*ge-36*co40)*lk*pi**2-co40*ge*lk*ll**2)*xz

ma(4,4) = (4*co20*ll**2*pi**2+(co28+1)*ge*lk**2*ll**2)*xy+16*co28*
1    pi**4+4*co20*ge*lk**2*pi**2

ma(4,5) = ge*lk**2*ll**2*xy

ma(4,6) = ge*lk**2*ll**2*xy

ma(4,7) = (co4*ge*lk**3*ll-2*co12*lk*ll*pi**2)*xy

ma(4,8) = (8*co28*lk*pi**3+2*co20*ge*lk**3*pi)*xz

ma(4,9) = (co29*ge*lk**3*ll-6*co23*lk*ll*pi**2)*xy
```

```
      ma(4,10) = co4*ge*lk*ll**2*xy+(2*co12*ge-4*co4)*lk*pi**2

      ma(4,11) = ((2*co28+2*co20)*ge+2*co20)*lk*ll*pi*xz

      ma(4,12) = co29*ge*lk*ll**2*xy+(6*co23*ge-4*co29)*lk*pi**2

      ma(5,5) = (16*co46*ll**2*pi**2+(co50+1)*ge*lk**2*ll**2)*xy+256*co5
     1   0*pi**4+16*co46*ge*lk**2*pi**2

      ma(5,6) = ge*lk**2*ll**2*xy

      ma(5,7) = (co8*ge*lk**3*ll-4*co16*lk*ll*pi**2)*xy

      ma(5,8) = 0

      ma(5,9) = (co38*ge*lk**3*ll-12*co42*lk*ll*pi**2)*xy

      ma(5,10) = co8*ge*lk*ll**2*xy+(4*co16*ge-16*co8)*lk*pi**2

      ma(5,11) = 0

      ma(5,12) = co38*ge*lk*ll**2*xy+(12*co42*ge-16*co38)*lk*pi**2

      ma(6,6) = (36*co53*ll**2*pi**2+(co55+1)*ge*lk**2*ll**2)*xy+1296*co
     1   55*pi**4+36*co53*ge*lk**2*pi**2

      ma(6,7) = (co10*ge*lk**3*ll-6*co18*lk*ll*pi**2)*xy

      ma(6,8) = 0

      ma(6,9) = (co40*ge*lk**3*ll-18*co44*lk*ll*pi**2)*xy

      ma(6,10) = co10*ge*lk*ll**2*xy+(6*co18*ge-36*co10)*lk*pi**2

      ma(6,11) = 0

      ma(6,12) = co40*ge*lk*ll**2*xy+(18*co44*ge-36*co40)*lk*pi**2

      ma(7,7) = (co11*lk**2*pi**2+co1*ge*lk**4)*xy+co11*la*pi**2

      ma(7,8) = 0

      ma(7,9) = 0

      ma(7,10) = co1*ge*lk**2*ll*xy
```

```
      ma(7,11) = (2*co4*ge-co12)*lk**2*pi*xz

      ma(7,12) = 0

      ma(8,8) = (4*co28*lk**2*pi**2+co20*ge*lk**4)*xy+4*co28*la*pi**2

      ma(8,9) = 0

      ma(8,10) = (co12*ge-2*co4)*lk**2*pi*xz

      ma(8,11) = co20*ge*lk**2*ll*xy

      ma(8,12) = (3*co23*ge-2*co29)*lk**2*pi*xz

      ma(9,9) = (9*co41*lk**2*pi**2+co35*ge*lk**4)*xy+9*co41*la*pi**2

      ma(9,10) = 0

      ma(9,11) = (2*co29*ge-3*co23)*lk**2*pi*xz

      ma(9,12) = co35*ge*lk**2*ll*xy

      ma(10,10) = -lmda*(-co1*xy-co1)+(co11*ge*pi**2-co1*lmda+co1*ge*ll*
     1   *2)*xy+co11*ge*pi**2-co1*lmda+co1*ge*ll**2+co1*lk**2

      ma(10,11) = 0

      ma(10,12) = 0

      ma(11,11) = -lmda*(-co20*xy-co20)+(4*co28*ge*pi**2-co20*lmda+co20*
     1   ge*ll**2)*xy+4*co28*ge*pi**2-co20*lmda+co20*ge*ll**2+co20*lk**2

      ma(11,12) = 0

      ma(12,12) = -lmda*(-co35*xy-co35)+(9*co41*ge*pi**2-co35*lmda+co35*
     1   ge*ll**2)*xy+9*co41*ge*pi**2-co35*lmda+co35*ge*ll**2+co35*lk**2
```

B.2 Matrix Mb

$$\mathbf{M}_b(i,j) = \mathbf{M}_b(j,i) \quad for \quad i \neq j \quad (i,j = 1.....12)$$

```
mb(1,1) = -(-co28-1)*la

mb(1,2) = la

mb(1,3) = la

mb(1,4) = 0

mb(1,5) = 0

mb(1,6) = 0

mb(1,7) = 0

mb(1,8) = 0

mb(1,9) = 0

mb(1,10) = 0

mb(1,11) = 0

mb(1,12) = 0

mb(2,2) = -(-co50-1)*la

mb(2,3) = la

mb(2,4) = 0

mb(2,5) = 0

mb(2,6) = 0

mb(2,7) = 0

mb(2,8) = 0

mb(2,9) = 0
```

```
mb(2,10) = 0

mb(2,11) = 0

mb(2,12) = 0

mb(3,3) = -(-co55-1)*la

mb(3,4) = 0

mb(3,5) = 0

mb(3,6) = 0

mb(3,7) = 0

mb(3,8) = 0

mb(3,9) = 0

mb(3,10) = 0

mb(3,11) = 0

mb(3,12) = 0

mb(4,4) = -(-co28-1)*la

mb(4,5) = la

mb(4,6) = la

mb(4,7) = 0

mb(4,8) = 0

mb(4,9) = 0

mb(4,10) = 0

mb(4,11) = 0

mb(4,12) = 0

mb(5,5) = -(-co50-1)*la
```

```
mb(5,6) = la
mb(5,7) = 0
mb(5,8) = 0
mb(5,9) = 0
mb(5,10) = 0
mb(5,11) = 0
mb(5,12) = 0
mb(6,6) = -(-co55-1)*la
mb(6,7) = 0
mb(6,8) = 0
mb(6,9) = 0
mb(6,10) = 0
mb(6,11) = 0
mb(6,12) = 0
mb(7,7) = co1*la
mb(7,8) = 0
mb(7,9) = 0
mb(7,10) = 0
mb(7,11) = 0
mb(7,12) = 0
mb(8,8) = co20*la
mb(8,9) = 0
mb(8,10) = 0
```

```
mb(8,11) = 0

mb(8,12) = 0

mb(9,9) = co35*la

mb(9,10) = 0

mb(9,11) = 0

mb(9,12) = 0

mb(10,10) = co1*xy+co1

mb(10,11) = 0

mb(10,12) = 0

mb(11,11) = co20*xy+co20

mb(11,12) = 0

mb(12,12) = co35*xy+co35
```

B.3 Integrated Coefficients

The integrated coefficients. co1 to co55. shown in matrices M_a and M_b are the results of the product of functions ξ and η integrated from $z = 0$ to $z = l$.

Functions ξ and η as well as the relations between co$n$ (n = 1.....55) and functions ξ, η are given in table B-1.

From table B-1, we arrive at the following values.

```
co1=    0.5
co2=    0
co3=    0
```

ε	η									
	sin(t1)	cos(t1)	sin(t2)	cos(t2)	sin(t3)	cos(t3)	sin(t4)	cos(t4)	sin(t5)	cos(t5)
sin(t1)	co1									
cos(t1)	co2	co11								
sin(t2)	co3	co12	co20							
cos(t2)	co4	co13	co21	co28						
sin(t3)	co5	co14	co22	co29	co35					
cos(t3)	co6	co15	co23	co30	co36	co41				
sin(t4)	co7	co16	co24	co31	co37	co42	co46			
cos(t4)	co8	co17	co25	co32	co38	co43	co47	co50		
sin(t5)	co9	co18	co26	co33	co39	co44	co48	co51	co53	
cos(t5)	co10	co19	co27	co34	co40	co45	co49	co52	co54	co55

Note: $t1=\pi z/l$ $t2=2\pi z/l$ $t3=3\pi z/l$ $t4=4\pi z/l$ $t5=6\pi z/l$

Table B-1: Functions ξ and η (Clamped-Clamped, Admissible Functions)

co4= - 0.2122065907891938
co5= 0
co6= 0
co7= 0
co8= - 0.04244131815783876
co9= 0
co10=- 0.01818913635335947
co11= 0.5
co12= 0.4244131815783876
co13= 0
co14= 0
co15= 0
co16= 0.169765272631355
co17= 0
co18= 0.1091348181201568
co19= 0
co20= 0.5
co21= 0
co22= 0
co23=- 0.2546479089470326
co24= 0
co25= 0
co26= 0
co27= 0
co28= 0.5
co29= 0.3819718634205488
co30= 0
co31= 0
co32= 0
co33= 0
co34= 0
co35= 0.5
co36= 0
co37= 0
co38=- 0.272837045300392
co39= 0
co40=- 0.0707355302630646
co41= 0.5
co42= 0.3637827270671894
co43= 0
co44= 0.1414710605261292
co45= 0
co46= 0.5
co47= 0
co48= 0

```
co49=  0
co50=  0.5
co51=  0
co52=  0
co53=  0.5
co54=  0
co55=  0.5
```

APPENDIX C

MATRICES MC AND MD

C.1 Matrix Mc

$$\mathbf{M}_c\,(i,j) = \mathbf{M}_c\,(j,i) \quad for \quad i \neq j \quad (i,j = 1,\ldots,8)$$

```
 mc(1,1) = (((2*co70-2*co65-2*co61+2*co59)*ge*k1*kk1**2+(-2*co73-2*
1    co69+4*co66-2*co64+2*co58)*ge*k1*kk1+(2*co70-2*co65+2*co61-2*co
2    59)*ge*k1)*lk**2+(2*co70-2*co65+2*co61-2*co59)*k1**3*kk1**2+(-2
3    *co73-2*co69+2*co64-4*co60-2*co58)*k1**3*kk1+(2*co70+2*co65+2*c
4    o61+2*co59)*k1**3)*ll*xz+(((co73-2*co66+co64)*ge*k1**2*kk1**2+(
5    -2*co70+2*co65-2*co61+2*co59)*ge*k1**2*kk1+(co69+2*co60+co58)*g
6    e*k1**2)*lk**2+(co69+2*co60+co58)*k1**4*kk1**2+(-2*co70-2*co65-
7    2*co61-2*co59)*k1**4*kk1+(co73+2*co66+co64)*k1**4)*xy+(((co69-2
8    *co60+co58)*ge*kk1**2+(-2*co70+2*co65+2*co61-2*co59)*ge*kk1+(co
9    73-2*co66+co64)*ge)*lk**2+(co73-2*co66+co64)*k1**2*kk1**2+(-2*c
:    o70+2*co65-2*co61+2*co59)*k1**2*kk1+(co69+2*co60+co58)*k1**2)*l
;    1**2+((co69-2*co60+co58)*kk1**2+(-2*co70+2*co65+2*co61-2*co59)*
<    kk1+co73-2*co66+co64)*la*lk**2

 mc(1,2) = (((((co54-co52-co39+co37)*ge*k2+(co48-co46-co31+co29)*ge
1    *k1)*kk1+(-co55+co53+co40-co38)*ge*k2+(-co47-co45+co30+co28)*ge
2    *k1)*kk2+((-co47+co45-co30+co28)*ge*k2+(-co55+co53+co40-co38)*g
3    e*k1)*kk1+(co48-co46+co31-co29)*ge*k2+(co54+co52-co39-co37)*ge*
4    k1)*lk**2+(((co48-co46+co31-co29)*k1*k2**2+(co54+co52-co39-co37
5    )*k1**2*k2)*kk1+(-co47-co45-co30-co28)*k1*k2**2+(-co55-co53+co4
6    0+co38)*k1**2*k2)*kk2+((-co55+co53-co40+co38)*k1*k2**2+(-co47-c
7    o45-co30-co28)*k1**2*k2)*kk1+(co54+co52+co39+co37)*k1*k2**2+(co
8    48+co46+co31+co29)*k1**2*k2)*ll*xz+((((co55-co53-co40+co38)*ge*
9    k1*k2*kk1+(-co54-co52+co39+co37)*ge*k1*k2)*kk2+(-co48+co46-co31
:    +co29)*ge*k1*k2*kk1+(co47+co45+co30+co28)*ge*k1*k2)*lk**2+((co4
;    7+co45+co30+co28)*k1**2*k2**2*kk1+(-co48-co46-co31-co29)*k1**2*
<    k2**2)*kk2+(-co54-co52-co39-co37)*k1**2*k2**2*kk1+(co55+co53+co
=    40+co38)*k1**2*k2**2)*xy+((((co47-co45-co30+co28)*ge*kk1+(-co48
>    +co46+co31-co29)*ge)*kk2+(-co54+co52+co39-co37)*ge*kk1+(co55-co
?    53-co40+co38)*ge)*lk**2+((co55-co53-co40+co38)*k1*k2*kk1+(-co54
```

```
     -co52+co39+co37)*k1*k2)*kk2+(-co48+co46-co31+co29)*k1*k2*kk1+(c
     1 o47+co45+co30+co28)*k1*k2)*ll**2+(((co47-co45-co30+co28)*kk1-co
     2 48+co46+co31-co29)*kk2+(-co54+co52+co39-co37)*kk1+co55-co53-co4
     3 0+co38)*la*lk**2

      mc(1,3) = ((((-co69+2*co60-co58)*ge*kk1**2+(2*co70-2*co65-2*co61+2
     1 *co59)*ge*kk1+(-co73+2*co66-co64)*ge)*lk**2+(-co73+2*co66-co64)
     2 *k1**2*kk1**2+(2*co70-2*co65+2*co61-2*co59)*k1**2*kk1+(-co69-2*
     3 co60-co58)*k1**2)*ll**2+((co73-2*co66+co64)*ge*k1**2*kk1**2+(-2
     4 *co70+2*co65-2*co61+2*co59)*ge*k1**2*kk1+(co69+2*co60+co58)*ge*
     5 k1**2)*lk**2+(co69+2*co60+co58)*k1**4*kk1**2+(-2*co70-2*co65-2*
     6 co61-2*co59)*k1**4*kk1+(co73+2*co66+co64)*k1**4)*xz+(((-co70+co
     7 65+co61-co59)*ge*k1*kk1**2+(co73+co69-2*co66+co64-co58)*ge*k1*k
     8 k1+(-co70+co65-co61+co59)*ge*k1)*lk**2+(-co70+co65-co61+co59)*k
     9 1**3*kk1**2+(co73+co69-co64+2*co60+co58)*k1**3*kk1+(-co70-co65-
     : co61-co59)*k1**3)*ll*xy+(((co70-co65-co61+co59)*ge*k1*kk1**2+(-
     ; co73-co69+2*co66-co64+co58)*ge*k1*kk1+(co70-co65+co61-co59)*ge*
     < k1)*lk**2+(co70-co65+co61-co59)*k1**3*kk1**2+(-co73-co69+co64-2
     = *co60-co58)*k1**3*kk1+(co70+co65+co61+co59)*k1**3)*ll

      mc(1,4) = (((((-co47+co45+co30-co28)*ge*kk1+(co48-co46-co31+co29)*
     1 ge)*kk2+(co54-co52-co39+co37)*ge*kk1+(-co55+co53+co40-co38)*ge)
     2 *lk**2+((-co55+co53+co40-co38)*k1*k2*kk1+(co54+co52-co39-co37)*
     3 k1*k2)*kk2+(co48-co46+co31-co29)*k1*k2*kk1+(-co47-co45-co30-co2
     4 8)*k1*k2)*ll**2+(((co55-co53-co40+co38)*ge*k1*k2*kk1+(-co54-co5
     5 2+co39+co37)*ge*k1*k2)*kk2+(-co48+co46-co31+co29)*ge*k1*k2*kk1+
     6 (co47+co45+co30+co28)*ge*k1*k2)*lk**2+((co47+co45+co30+co28)*k1
     7 **2*k2**2*kk1+(-co48-co46-co31-co29)*k1**2*k2**2)*kk2+(-co54-co
     8 52-co39-co37)*k1**2*k2**2*kk1+(co55+co53+co40+co38)*k1**2*k2**2
     9 )*xz+((((-co48+co46+co31-co29)*ge*k1*kk1+(co47+co45-co30-co28)*
     : ge*k1)*kk2+(co55-co53-co40+co38)*ge*k1*kk1+(-co54-co52+co39+co3
     ; 7)*ge*k1)*lk**2+((-co54-co52+co39+co37)*k1**2*k2*kk1+(co55+co53
     < -co40-co38)*k1**2*k2)*kk2+(co47+co45+co30+co28)*k1**2*k2*kk1+(-
     = co48-co46-co31-co29)*k1**2*k2)*ll*xy+((((co54-co52-co39+co37)*g
     > e*k2*kk1+(-co55+co53+co40-co38)*ge*k2)*kk2+(-co47+co45-co30+co2
     ? 8)*ge*k2*kk1+(co48-co46+co31-co29)*ge*k2)*lk**2+((co48-co46+co3
       1-co29)*k1*k2**2*kk1+(-co47-co45-co30-co28)*k1*k2**2)*kk2+(-co5
     1 5+co53-co40+co38)*k1*k2**2*kk1+(co54+co52+co39+co37)*k1*k2**2)*
     2 ll

      mc(1,5) = (((co68-co75)*k1*kk1+(co72+co63)*k1)*lk*ll*pi+((co62-co7
     1 1)*ge*kk1+(co74-co67)*ge)*lk**3*ll)*xz+(((-co72-co63)*k1**2*kk1
     2 +(co75+co68)*k1**2)*lk*pi+((co67-co74)*ge*k1*kk1+(co71+co62)*ge
     3 *k1)*lk**3)*xy+((co72-co63)*kk1-co75+co68)*la*lk*pi
```

```
mc(1,6) = (((2*co19-2*co21)*k1*kk1+(2*co20+2*co18)*k1)*lk*ll*pi+((
1    co7-co9)*ge*kk1+(co10-co8)*ge)*lk**3*ll)*xz+(((-2*co20-2*co18)*
2    k1**2*kk1+(2*co21+2*co19)*k1**2)*lk*pi+((co8-co10)*ge*k1*kk1+(c
3    o9+co7)*ge*k1)*lk**3)*xy+((2*co20-2*co18)*kk1-2*co21+2*co19)*la
4    *lk*pi

mc(1,7) = (((co68-co75)*ge*k1*kk1+(co72+co63)*ge*k1)*lk*pi+((co62-
1    co71)*ge*kk1+(co74-co67)*ge)*lk*ll**2+((co71+co62)*k1**2*kk1+(-
2    co74-co67)*k1**2)*lk)*xz+((co67-co74)*ge*k1*kk1+(co71+co62)*ge*
3    k1)*lk*ll*xy+((co63-co72)*ge*kk1+(co75-co68)*ge)*lk*ll*pi+((co7
4    4-co67)*k1*kk1+(-co71-co62)*k1)*lk*ll

mc(1,8) = (((2*co19-2*co21)*ge*k1*kk1+(2*co20+2*co18)*ge*k1)*lk*pi
1    +((co7-co9)*ge*kk1+(co10-co8)*ge)*lk*ll**2+((co9+co7)*k1**2*kk1
2    +(-co8-co10)*k1**2)*lk)*xz+((co8-co10)*ge*k1*kk1+(co9+co7)*ge*k
3    1)*lk*ll*xy+((2*co18-2*co20)*ge*kk1+(2*co21-2*co19)*ge)*lk*ll*p
4    i+((co10-co8)*k1*kk1+(-co9-co7)*k1)*lk*ll

mc(2,2) = (((2*co44-2*co35-2*co27+2*co25)*ge*k2*kk2**2+(-2*co51-2*
1    co43+4*co36-2*co34+2*co24)*ge*k2*kk2+(2*co44-2*co35+2*co27-2*co
2    25)*ge*k2)*lk**2+(2*co44-2*co35+2*co27-2*co25)*k2**3*kk2**2+(-2
3    *co51-2*co43+2*co34-4*co26-2*co24)*k2**3*kk2+(2*co44+2*co35+2*c
4    o27+2*co25)*k2**3)*ll*xz+(((co51-2*co36+co34)*ge*k2**2*kk2**2+(
5    -2*co44+2*co35-2*co27+2*co25)*ge*k2**2*kk2+(co43+2*co26+co24)*g
6    e*k2**2)*lk**2+(co43+2*co26+co24)*k2**4*kk2**2+(-2*co44-2*co35-
7    2*co27-2*co25)*k2**4*kk2+(co51+2*co36+co34)*k2**4)*xy+(((co43-2
8    *co26+co24)*ge*kk2**2+(-2*co44+2*co35+2*co27-2*co25)*ge*kk2+(co
9    51-2*co36+co34)*ge)*lk**2+(co51-2*co36+co34)*k2**2*kk2**2+(-2*c
:    o44+2*co35-2*co27+2*co25)*k2**2*kk2+(co43+2*co26+co24)*k2**2)*l
;    l**2+((co43-2*co26+co24)*kk2**2+(-2*co44+2*co35+2*co27-2*co25)*
<    kk2+co51-2*co36+co34)*la*lk**2

mc(2,3) = (((((-co47+co45+co30-co28)*ge*kk1+(co48-co46-co31+co29)*
1    ge)*kk2+(co54-co52-co39+co37)*ge*kk1+(-co55+co53+co40-co38)*ge)
2    *lk**2+((-co55+co53+co40-co38)*k1*k2*kk1+(co54+co52-co39-co37)*
3    k1*k2)*kk2+(co48-co46+co31-co29)*k1*k2*kk1+(-co47-co45-co30-co2
4    8)*k1*k2)*ll**2+(((co55-co53-co40+co38)*ge*k1*k2*kk1+(-co54-co5
5    2+co39+co37)*ge*k1*k2)*kk2+(-co48+co46-co31+co29)*ge*k1*k2*kk1+
6    (co47+co45+co30+co28)*ge*k1*k2)*lk**2+((co47+co45+co30+co28)*k1
7    **2*k2**2*kk1+(-co48-co46-co31-co29)*k1**2*k2**2)*kk2+(-co54-co
8    52-co39-co37)*k1**2*k2**2*kk1+(co55+co53+co40+co38)*k1**2*k2**2
9    )*xz+((((-co54+co52+co39-co37)*ge*k2*kk1+(co55-co53-co40+co38)*
:    ge*k2)*kk2+(co47-co45+co30-co28)*ge*k2*kk1+(-co48+co46-co31+co2
;    9)*ge*k2)*lk**2+((-co48+co46-co31+co29)*k1*k2**2*kk1+(co47+co45
<    +co30+co28)*k1*k2**2)*kk2+(co55-co53+co40-co38)*k1*k2**2*kk1+(-
```

```
=    co54-co52-co39-co37)*k1*k2**2)*ll*xy+((((co48-co46-co31+co29)*g
>    e*k1*kk1+(-co47-co45+co30+co28)*ge*k1)*kk2+(-co55+co53+co40-co3
?    8)*ge*k1*kk1+(co54+co52-co39-co37)*ge*k1)*lk**2+((co54+co52-co3
    9-co37)*k1**2*k2*kk1+(-co55-co53+co40+co38)*k1**2*k2)*kk2+(-co4
1    7-co45-co30-co28)*k1**2*k2*kk1+(co48+co46+co31+co29)*k1**2*k2)*
2    ll

mc(2,4) = ((((-co43+2*co26-co24)*ge*kk2**2+(2*co44-2*co35-2*co27+2
1    *co25)*ge*kk2+(-co51+2*co36-co34)*ge)*lk**2+(-co51+2*co36-co34)
2    *k2**2*kk2**2+(2*co44-2*co35+2*co27-2*co25)*k2**2*kk2+(-co43-2*
3    co26-co24)*k2**2)*ll**2+((co51-2*co36+co34)*ge*k2**2*kk2**2+(-2
4    *co44+2*co35-2*co27+2*co25)*ge*k2**2*kk2+(co43+2*co26+co24)*ge*
5    k2**2)*lk**2+(co43+2*co26+co24)*k2**4*kk2**2+(-2*co44-2*co35-2*
6    co27-2*co25)*k2**4*kk2+(co51+2*co36+co34)*k2**4)*xz+(((-co44+co
7    35+co27-co25)*ge*k2*kk2**2+(co51+co43-2*co36+co34-co24)*ge*k2*k
8    k2+(-co44+co35-co27+co25)*ge*k2)*lk**2+(-co44+co35-co27+co25)*k
9    2**3*kk2**2+(co51+co43-co34+2*co26+co24)*k2**3*kk2+(-co44-co35-
:    co27-co25)*k2**3)*ll*xy+(((co44-co35-co27+co25)*ge*k2*kk2**2+(-
;    co51-co43+2*co36-co34+co24)*ge*k2*kk2+(co44-co35+co27-co25)*ge*
<    k2)*lk**2+(co44-co35+co27-co25)*k2**3*kk2**2+(-co51-co43+co34-2
=    *co26-co24)*k2**3*kk2+(co44+co35+co27+co25)*k2**3)*ll

mc(2,5) = (((co42-co57)*k2*kk2+(co50+co33)*k2)*lk*ll*pi+((co32-co4
1    9)*ge*kk2+(co56-co41)*ge)*lk**3*ll)*xz+(((-co50-co33)*k2**2*kk2
2    +(co57+co42)*k2**2)*lk*pi+((co41-co56)*ge*k2*kk2+(co49+co32)*ge
3    *k2)*lk**3)*xy+((co50-co33)*kk2-co57+co42)*la*lk*pi

mc(2,6) = (((2*co15-2*co17)*k2*kk2+(2*co16+2*co14)*k2)*lk*ll*pi+((
1    co3-co5)*ge*kk2+(co6-co4)*ge)*lk**3*ll)*xz+(((-2*co16-2*co14)*k
2    2**2*kk2+(2*co17+2*co15)*k2**2)*lk*pi+((co4-co6)*ge*k2*kk2+(co5
3    +co3)*ge*k2)*lk**3)*xy+((2*co16-2*co14)*kk2-2*co17+2*co15)*la*l
4    k*pi

mc(2,7) = (((co42-co57)*ge*k2*kk2+(co50+co33)*ge*k2)*lk*pi+((co32-
1    co49)*ge*kk2+(co56-co41)*ge)*lk*ll**2+((co49+co32)*k2**2*kk2+(-
2    co56-co41)*k2**2)*lk)*xz+((co41-co56)*ge*k2*kk2+(co49+co32)*ge*
3    k2)*lk*ll*xy+((co33-co50)*ge*kk2+(co57-co42)*ge)*lk*ll*pi+((co5
4    6-co41)*k2*kk2+(-co49-co32)*k2)*lk*ll

mc(2,8) = (((2*co15-2*co17)*ge*k2*kk2+(2*co16+2*co14)*ge*k2)*lk*pi
1    +((co3-co5)*ge*kk2+(co6-co4)*ge)*lk*ll**2+((co5+co3)*k2**2*kk2+
2    (-co6-co4)*k2**2)*lk)*xz+((co4-co6)*ge*k2*kk2+(co5+co3)*ge*k2)*
3    lk*ll*xy+((2*co14-2*co16)*ge*kk2+(2*co17-2*co15)*ge)*lk*ll*pi+(
4    (co6-co4)*k2*kk2+(-co5-co3)*k2)*lk*ll
```

```
 mc(3,3) = (((-2*co70+2*co65+2*co61-2*co59)*ge*k1*kk1**2+(2*co73+2*
1    co69-4*co66+2*co64-2*co58)*ge*k1*kk1+(-2*co70+2*co65-2*co61+2*c
2    o59)*ge*k1)*lk**2+(-2*co70+2*co65-2*co61+2*co59)*k1**3*kk1**2+(
3    2*co73+2*co69-2*co64+4*co60+2*co58)*k1**3*kk1+(-2*co70-2*co65-2
4    *co61-2*co59)*k1**3)*ll*xz+(((co69-2*co60+co58)*ge*kk1**2+(-2*c
5    o70+2*co65+2*co61-2*co59)*ge*kk1+(co73-2*co66+co64)*ge)*lk**2+(
6    co73-2*co66+co64)*k1**2*kk1**2+(-2*co70+2*co65-2*co61+2*co59)*k
7    1**2*kk1+(co69+2*co60+co58)*k1**2)*ll**2*xy+((co73-2*co66+co64)
8    *ge*k1**2*kk1**2+(-2*co70+2*co65-2*co61+2*co59)*ge*k1**2*kk1+(c
9    o69+2*co60+co58)*ge*k1**2)*lk**2+(co69+2*co60+co58)*k1**4*kk1**
:    2+(-2*co70-2*co65-2*co61-2*co59)*k1**4*kk1+(co73+2*co66+co64)*k
;    1**4

 mc(3,4) = (((((-co54+co52+co39-co37)*ge*k2+(-co48+co46+co31-co29)*
1    ge*k1)*kk1+(co55-co53-co40+co38)*ge*k2+(co47+co45-co30-co28)*ge
2    *k1)*kk2+((co47-co45+co30-co28)*ge*k2+(co55-co53-co40+co38)*ge*
3    k1)*kk1+(-co48+co46-co31+co29)*ge*k2+(-co54-co52+co39+co37)*ge*
4    k1)*lk**2+(((-co48+co46-co31+co29)*k1*k2**2+(-co54-co52+co39+co
5    37)*k1**2*k2)*kk1+(co47+co45+co30+co28)*k1*k2**2+(co55+co53-co4
6    0-co38)*k1**2*k2)*kk2+((co55-co53+co40-co38)*k1*k2**2+(co47+co4
7    5+co30+co28)*k1**2*k2)*kk1+(-co54-co52-co39-co37)*k1*k2**2+(-co
8    48-co46-co31-co29)*k1**2*k2)*ll*xz+((((co47-co45-co30+co28)*ge*
9    kk1+(-co48+co46+co31-co29)*ge)*kk2+(-co54+co52+co39-co37)*ge*kk
:    1+(co55-co53-co40+co38)*ge)*lk**2+((co55-co53-co40+co38)*k1*k2*
;    kk1+(-co54-co52+co39+co37)*k1*k2)*kk2+(-co48+co46-co31+co29)*k1
<    *k2*kk1+(co47+co45+co30+co28)*k1*k2)*ll**2*xy+(((co55-co53-co40
=    +co38)*ge*k1*k2*kk1+(-co54-co52+co39+co37)*ge*k1*k2)*kk2+(-co48
>    +co46-co31+co29)*ge*k1*k2*kk1+(co47+co45+co30+co28)*ge*k1*k2)*l
?    k**2+((co47+co45+co30+co28)*k1**2*k2**2*kk1+(-co48-co46-co31-co
    29)*k1**2*k2**2)*kk2+(-co54-co52-co39-co37)*k1**2*k2**2*kk1+(co
1    55+co53+co40+co38)*k1**2*k2**2

 mc(3,5) = (((-co72-co63)*k1**2*kk1+(co75+co68)*k1**2)*lk*pi+((co67
1    -co74)*ge*k1*kk1+(co71+co62)*ge*k1)*lk**3)*xz+(((co75-co68)*k1*
2    kk1+(-co72-co63)*k1)*lk*ll*pi+((co71-co62)*ge*kk1+(co67-co74)*g
3    e)*lk**3*ll)*xy

 mc(3,6) = (((-2*co20-2*co18)*k1**2*kk1+(2*co21+2*co19)*k1**2)*lk*p
1    i+((co8-co10)*ge*k1*kk1+(co9+co7)*ge*k1)*lk**3)*xz+(((2*co21-2*
2    co19)*k1*kk1+(-2*co20-2*co18)*k1)*lk*ll*pi+((co9-co7)*ge*kk1+(c
3    o8-co10)*ge)*lk**3*ll)*xy

 mc(3,7) = (((co72-co63)*ge*kk1+(co68-co75)*ge)*lk*ll*pi+(((co67-co
1    74)*ge-co74+co67)*k1*kk1+((co71+co62)*ge+co71+co62)*k1)*lk*ll)*
2    xz+((co71-co62)*ge*kk1+(co67-co74)*ge)*lk*ll**2*xy+((co68-co75)
```

```
3    *ge*k1*kk1+(co72+co63)*ge*k1)*lk*pi+((co71+co62)*k1**2*kk1+(-co
4    74-co67)*k1**2)*lk

 mc(3,8) = (((2*co20-2*co18)*ge*kk1+(2*co19-2*co21)*ge)*lk*ll*pi+((
1    (co8-co10)*ge+co8-co10)*k1*kk1+((co9+co7)*ge+co9+co7)*k1)*lk*ll
2    )*xz+((co9-co7)*ge*kk1+(co8-co10)*ge)*lk*ll**2*xy+((2*co19-2*co
3    21)*ge*k1*kk1+(2*co20+2*co18)*ge*k1)*lk*pi+((co9+co7)*k1**2*kk1
4    +(-co8-co10)*k1**2)*lk

 mc(4,4) = (((-2*co44+2*co35+2*co27-2*co25)*ge*k2*kk2**2+(2*co51+2*
1    co43-4*co36+2*co34-2*co24)*ge*k2*kk2+(-2*co44+2*co35-2*co27+2*c
2    o25)*ge*k2)*lk**2+(-2*co44+2*co35-2*co27+2*co25)*k2**3*kk2**2+(
3    2*co51+2*co43-2*co34+4*co26+2*co24)*k2**3*kk2+(-2*co44-2*co35-2
4    *co27-2*co25)*k2**3)*ll*xz+(((co43-2*co26+co24)*ge*kk2**2+(-2*c
5    o44+2*co35+2*co27-2*co25)*ge*kk2+(co51-2*co36+co34)*ge)*lk**2+(
6    co51-2*co36+co34)*k2**2*kk2**2+(-2*co44+2*co35-2*co27+2*co25)*k
7    2**2*kk2+(co43+2*co26+co24)*k2**2)*ll**2*xy+((co51-2*co36+co34)
8    *ge*k2**2*kk2**2+(-2*co44+2*co35-2*co27+2*co25)*ge*k2**2*kk2+(c
9    o43+2*co26+co24)*ge*k2**2)*lk**2+(co43+2*co26+co24)*k2**4*kk2**
:    2+(-2*co44-2*co35-2*co27-2*co25)*k2**4*kk2+(co51+2*co36+co34)*k
;    2**4

 mc(4,5) = (((-co50-co33)*k2**2*kk2+(co57+co42)*k2**2)*lk*pi+((co41
1    -co56)*ge*k2*kk2+(co49+co32)*ge*k2)*lk**3)*xz+(((co57-co42)*k2*
2    kk2+(-co50-co33)*k2)*lk*ll*pi+((co49-co32)*ge*kk2+(co41-co56)*g
3    e)*lk**3*ll)*xy

 mc(4,6) = (((-2*co16-2*co14)*k2**2*kk2+(2*co17+2*co15)*k2**2)*lk*p
1    i+((co4-co6)*ge*k2*kk2+(co5+co3)*ge*k2)*lk**3)*xz+(((2*co17-2*c
2    o15)*k2*kk2+(-2*co16-2*co14)*k2)*lk*ll*pi+((co5-co3)*ge*kk2+(co
3    4-co6)*ge)*lk**3*ll)*xy

 mc(4,7) = (((co50-co33)*ge*kk2+(co42-co57)*ge)*lk*ll*pi+(((co41-co
1    56)*ge-co56+co41)*k2*kk2+((co49+co32)*ge+co49+co32)*k2)*lk*ll)*
2    xz+((co49-co32)*ge*kk2+(co41-co56)*ge)*lk*ll**2*xy+((co42-co57)
3    *ge*k2*kk2+(co50+co33)*ge*k2)*lk*pi+((co49+co32)*k2**2*kk2+(-co
4    56-co41)*k2**2)*lk

 mc(4,8) = (((2*co16-2*co14)*ge*kk2+(2*co15-2*co17)*ge)*lk*ll*pi+((
1    (co4-co6)*ge-co6+co4)*k2*kk2+((co5+co3)*ge+co5+co3)*k2)*lk*ll)*
2    xz+((co5-co3)*ge*kk2+(co4-co6)*ge)*lk*ll**2*xy+((2*co15-2*co17)
3    *ge*k2*kk2+(2*co16+2*co14)*ge*k2)*lk*pi+((co5+co3)*k2**2*kk2+(-
4    co6-co4)*k2**2)*lk

 mc(5,5) = (co78*lk**2*pi**2+co76*ge*lk**4)*xy+co78*la*pi**2
```

```
mc(5,6) = (2*co23*lk**2*pi**2+co11*ge*lk**4)*xy+2*co23*la*pi**2

mc(5,7) = (co77*ge-co77)*lk**2*pi*xz+co76*ge*lk**2*ll*xy

mc(5,8) = (2*co22*ge-co12)*lk**2*pi*xz+co11*ge*lk**2*ll*xy

mc(6,6) = (4*co13*lk**2*pi**2+co1*ge*lk**4)*xy+4*co13*la*pi**2

mc(6,7) = (co12*ge-2*co22)*lk**2*pi*xz+co11*ge*lk**2*ll*xy

mc(6,8) = (2*co2*ge-2*co2)*lk**2*pi*xz+co1*ge*lk**2*ll*xy

mc(7,7) = -lmda*(-co76*xy-co76)+(co78*ge*pi**2-co76*lmda+co76*ge*l
1    l**2)*xy+co78*ge*pi**2-co76*lmda+co76*ge*ll**2+co76*lk**2

mc(7,8) = -lmda*(-co11*xy-co11)+(2*co23*ge*pi**2-co11*lmda+co11*ge
1    *ll**2)*xy+2*co23*ge*pi**2-co11*lmda+co11*ge*ll**2+co11*lk**2

mc(8,8) = -lmda*(-co1*xy-co1)+(4*co13*ge*pi**2-co1*lmda+co1*ge*ll*
1    *2)*xy+4*co13*ge*pi**2-co1*lmda+co1*ge*ll**2+co1*lk**2
```

C.2 Matrix Md

$$\mathbf{M}_d(i,j) = \mathbf{M}_d(j,i) \quad for \quad i \neq j \quad (i,j = 1.....12)$$

```
md(1,1) = -((-co69+2*co60-co58)*kk1**2+(2*co70-2*co65-2*co61+2*co5
1    9)*kk1-co73+2*co66-co64)*la

md(1,2) = -((((-co47+co45+co30-co28)*kk1+co48-co46-co31+co29)*kk2+(
1    co54-co52-co39+co37)*kk1-co55+co53+co40-co38)*la

md(1,3) = 0

md(1,4) = 0

md(1,5) = 0

md(1,6) = 0

md(1,7) = 0

md(1,8) = 0
```

```
      md(2,2) = -((-co43+2*co26-co24)*kk2**2+(2*co44-2*co35-2*co27+2*co2
     1   5)*kk2-co51+2*co36-co34)*la

      md(2,3) = 0

      md(2,4) = 0

      md(2,5) = 0

      md(2,6) = 0

      md(2,7) = 0

      md(2,8) = 0

      md(3,3) = -((-co69+2*co60-co58)*kk1**2+(2*co70-2*co65-2*co61+2*co5
     1   9)*kk1-co73+2*co66-co64)*la

      md(3,4) = -(((-co47+co45+co30-co28)*kk1+co48-co46-co31+co29)*kk2+(
     1   co54-co52-co39+co37)*kk1-co55+co53+co40-co38)*la

      md(3,5) = 0

      md(3,6) = 0

      md(3,7) = 0

      md(3,8) = 0

      md(4,4) = -((-co43+2*co26-co24)*kk2**2+(2*co44-2*co35-2*co27+2*co2
     1   5)*kk2-co51+2*co36-co34)*la

      md(4,5) = 0

      md(4,6) = 0

      md(4,7) = 0

      md(4,8) = 0

      md(5,5) = co76*la

      md(5,6) = co11*la

      md(5,7) = 0
```

```
md(5,8) = 0

md(6,6) = co1*la

md(6,7) = 0

md(6,8) = 0

md(7,7) = co76*xy+co76

md(7,8) = co11*xy+co11

md(8,8) = co1*xy+co1
```

C.3 Integrated Coefficients

The integrated coefficients, co1 to co78, shown in matrices M_c and M_d are the results of the product of functions ξ and η integrated from $z = 0$ to $z = l$.

Functions ξ and η as well as the relations between co$n$ ($n = 1,\ldots,78$) and functions ξ, η are given in table C-1.

From table C-1, we arrive at the following values.

```
kk1=0.9825022145762
kk2=1.000777311907
k1=4.730040744863
k2=7.853204624096
co1=0.5
co2=0
co3=0.2830976246421625
co4=- 0.2828777404062893
co5=- 79.9432918768421
co6=- 79.88119926948301
co7=0.3672702673507307
co8=0.3608438510199932
co9=- 5.754332755625593
```

ξ	η											
	sin(t1)	cos(t1)	sin(t2)	cos(t2)	sinh(t2)	cosh(t2)	sin(t3)	cos(t3)	sinh(t3)	cosh(t3)	sin(t4)	cos(t4)
sin(t1)	co1											
cos(t1)	co2	co13										
sin(t2)	co3	co14	co24									
cos(t2)	co4	co15	co25	co34								
sinh(t2)	co5	co16	co26	co35	co43							
cosh(t2)	co6	co17	co27	co36	co44	co51						
sin(t3)	co7	co18	co28	co37	co45	co52	co58					
cos(t3)	co8	co19	co29	co38	co46	co53	co59	co64				
sinh(t3)	co9	co20	co30	co39	co47	co54	co60	co65	co69			
cosh(t3)	co10	co21	co31	co40	co48	co55	co61	co66	co70	co73		
sin(t4)	co11	co22	co32	co41	co49	co56	co62	co67	co71	co74	co76	
cos(t4)	co12	co23	co33	co42	co50	co57	co63	co68	co72	co75	co77	co78

Note: t1 = 2π z / l t2 = k2 z / l

t3 = k1 z / l t4 = π z / l

k1 : 4.7300 k2 : 7.8532

Table C-1: Functions ξ and η
(Clamped–Clamped, Normal Mode Functions)

co10=- 5.653644675810762
co11=0
co12=0.4244131815783876
co13=0.5
co14=0.353562193442563
co15=0.3538370215454286
co16=99.84162058130237
co17=99.91922866182124
co18=- 0.2716466305565636
co19=0.2764844969574014
co20=4.256116661561058
co21=4.331915591047025
co22=- 0.2122065907891938
co23=0
c024=0.4999505291404635
co25=0.0636682376254064
co26=81.87642671677981
co27=81.94007023819137
co28=0.002279677482065308
co29=0.3201666018578289
co30=3.184351735522682
co31=3.277293610756727
co32=- 0.0606447205059625
co33=0.151714640009954
co34=0.5000494708595365
co35=81.94007023816478
co36=82.00376323044337
co37=- 0.3201552871859263
co38=0.003620663312674233
co39=5.239125787449306
co40=5.296229223177236
co41=- 0.06069186036931498
co42=- 0.1515968020106723
co43=105455.1775540691
co44=105455.6457199359
co45=- 121.5146304049034
co46=- 70.38991464795974
co47=5792.223435488106
co48=5795.660646120006
co49=56.51445896834048
co50=- 141.3819822435741
co51=105456.1775540691
co52=- 121.4583153781003
co53=- 70.29649790838054
co54=5792.336747342393

```
co55=5795.853428456008
co56=56.55838833022944
co57=- 141.2721697038976
co58=0.5018655334042408
co59=0.105674401288916
co60=- 6.093554325721313
co61=- 5.986930619661868
co62=0.2512142798792595
co63=0.3849690308126355
co64=0.4981344665957592
co65=- 5.986930619658262
co66=- 5.88217259232853
co67=- 0.2556882581559216
co68=0.3782329253165705
co69=338.7394246092603
co70=339.1865750592409
co71=5.519283205241623
co72=- 8.457931242296016
co73=339.7394246092603
co74=5.617578386448839
co75=- 8.309936176289431
co76=0.5
co77=0
co78=0.5
```

APPENDIX D

MATRICES ME AND MF

D.1 Matrix Me

$$\mathbf{M}_e(i,j) = \mathbf{M}_e(j,i) \quad for \quad i \neq j \quad (i,j = 1,\ldots,8)$$

```
 me(1,1) = (((2*co70-2*co65-2*co61+2*co59)*ge*k1*kk1**2+(-2*co73-2*
1     co69+4*co66-2*co64+2*co58)*ge*k1*kk1+(2*co70-2*co65+2*co61-2*co
2     59)*ge*k1)*lk**2+(2*co70-2*co65+2*co61-2*co59)*k1**3*kk1**2+(-2
3     *co73-2*co69+2*co64-4*co60-2*co58)*k1**3*kk1+(2*co70+2*co65+2*c
4     o61+2*co59)*k1**3)*ll*xz+(((co73-2*co66+co64)*ge*k1**2*kk1**2+(
5     -2*co70+2*co65-2*co61+2*co59)*ge*k1**2*kk1+(co69+2*co60+co58)*g
6     e*k1**2)*lk**2+(co69+2*co60+co58)*k1**4*kk1**2+(-2*co70-2*co65-
7     2*co61-2*co59)*k1**4*kk1+(co73+2*co66+co64)*k1**4)*xy+(((co69-2
8     *co60+co58)*ge*kk1**2+(-2*co70+2*co65+2*co61-2*co59)*ge*kk1+(co
9     73-2*co66+co64)*ge)*lk**2+(co73-2*co66+co64)*k1**2*kk1**2+(-2*c
:     o70+2*co65-2*co61+2*co59)*k1**2*kk1+(co69+2*co60+co58)*k1**2)*l
;     l**2+((co69-2*co60+co58)*kk1**2+(-2*co70+2*co65+2*co61-2*co59)*
<     kk1+co73-2*co66+co64)*la*lk**2

 me(1,2) = (((((co54-co52-co39+co37)*ge*k2+(co48-co46-co31+co29)*ge
1     *k1)*kk1+(-co55+co53+co40-co38)*ge*k2+(-co47-co45+co30+co28)*ge
2     *k1)*kk2+((-co47[illegible]co45-co30+co28)*ge*k2+(-co55+co53+co40-co38)*g
3     e*k1)*kk1+(co48-c[illegible]+co31-co29)*ge*k2+(co54+co52-co39-co37)*ge*
4     k1)*lk**2+(((co48-co46+co31-co29)*k1*k2**2+(co54+co52-co39-co37
5     )*k1**2*k2)*kk1+(-co47-co45-co30-co28)*k1*k2**2+(-co55-co53+co4
6     0+co38)*k1**2*k2)*kk2+((-co55+co53-co40+co38)*k1*k2**2+(-co47-c
7     o45-co30-co28)*k1**2*k2)*kk1+(co54+co52+co39+co37)*k1*k2**2+(co
8     48+co46+co31+co29)*k1**2*k2)*ll*xz+((((co55-co53-co40+co38)*ge*
9     k1*k2*kk1+(-co54-co52+co39+co37)*ge*k1*k2)*kk2+(-co48+co46-co31
:     +co29)*ge*k1*k2*kk1+(co47+co45+co30+co28)*ge*k1*k2)*lk**2+((co4
;     7+co45+co30+co28)*k1**2*k2**2*kk1+(-co48-co46-co31-co29)*k1**2*
<     k2**2)*kk2+(-co54-co52-co39-co37)*k1**2*k2**2*kk1+(co55+co53+co
=     40+co38)*k1**2*k2**2)*xy+((((co47-co45-co30+co28)*ge*kk1+(-co48
>     +co46+co31-co29)*ge)*kk2+(-co54+co52+co39-co37)*ge*kk1+(co55-co
?     53-co40+co38)*ge)*lk**2+((co55-co53-co40+co38)*k1*k2*kk1+(-co54
```

```
   -co52+co39+co37)*k1*k2)*kk2+(-co48+co46-co31+co29)*k1*k2*kk1+(c
1    o47+co45+co30+co28)*k1*k2)*ll**2+(((co47-co45-co30+co28)*kk1-co
2    48+co46+co31-co29)*kk2+(-co54+co52+co39-co37)*kk1+co55-co53-co4
3    0+co38)*la*lk**2

 me(1,3) = ((((-co69+2*co60-co58)*ge*kk1**2+(2*co70-2*co65-2*co61+2
1    *co59)*ge*kk1+(-co73+2*co66-co64)*ge)*lk**2+(-co73+2*co66-co64)
2    *k1**2*kk1**2+(2*co70-2*co65+2*co61-2*co59)*k1**2*kk1+(-co69-2*
3    co60-co58)*k1**2)*ll**2+((co73-2*co66+co64)*ge*k1**2*kk1**2+(-2
4    *co70+2*co65-2*co61+2*co59)*ge*k1**2*kk1+(co69+2*co60+co58)*ge*
5    k1**2)*lk**2+(co69+2*co60+co58)*k1**4*kk1**2+(-2*co70-2*co65-2*
6    co61-2*co59)*k1**4*kk1+(co73+2*co66+co64)*k1**4)*xz+(((-co70+co
7    65+co61-co59)*ge*k1*kk1**2+(co73+co69-2*co66+co64-co58)*ge*k1*k
8    k1+(-co70+co65-co61+co59)*ge*k1)*lk**2+(-co70+co65-co61+co59)*k
9    1**3*kk1**2+(co73+co69-co64+2*co60+co58)*k1**3*kk1+(-co70-co65-
:    co61-co59)*k1**3)*ll*xy+(((co70-co65-co61+co59)*ge*k1*kk1**2+(-
;    co73-co69+2*co66-co64+co58)*ge*k1*kk1+(co70-co65+co61-co59)*ge*
<    k1)*lk**2+(co70-co65+co61-co59)*k1**3*kk1**2+(-co73-co69+co64-2
=    *co60-co58)*k1**3*kk1+(co70+co65+co61+co59)*k1**3)*ll

 me(1,4) = (((((-co47+co45+co30-co28)*ge*kk1+(co48-co46-co31+co29)*
1    ge)*kk2+(co54-co52-co39+co37)*ge*kk1+(-co55+co53+co40-co38)*ge)
2    *lk**2+((-co55+co53+co40-co38)*k1*k2*kk1+(co54+co52-co39-co37)*
3    k1*k2)*kk2+(co48-co46+co31-co29)*k1*k2*kk1+(-co47-co45-co30-co2
4    8)*k1*k2)*ll**2+(((co55-co53-co40+co38)*ge*k1*k2*kk1+(-co54-co5
5    2+co39+co37)*ge*k1*k2)*kk2+(-co48+co46-co31+co29)*ge*k1*k2*kk1+
6    (co47+co45+co30+co28)*ge*k1*k2)*lk**2+((co47+co45+co30+co28)*k1
7    **2*k2**2*kk1+(-co48-co46-co31-co29)*k1**2*k2**2)*kk2+(-co54-co
8    52-co39-co37)*k1**2*k2**2*kk1+(co55+co53+co40+co38)*k1**2*k2**2
9    )*xz+((((-co48+co46+co31-co29)*ge*k1*kk1+(co47+co45-co30-co28)*
:    ge*k1)*kk2+(co55-co53-co40+co38)*ge*k1*kk1+(-co54-co52+co39+co3
;    7)*ge*k1)*lk**2+((-co54-co52+co39+co37)*k1**2*k2*kk1+(co55+co53
<    -co40-co38)*k1**2*k2)*kk2+(co47+co45+co30+co28)*k1**2*k2*kk1+(-
=    co48-co46-co31-co29)*k1**2*k2)*ll*xy+((((co54-co52-co39+co37)*g
>    e*k2*kk1+(-co55+co53+co40-co38)*ge*k2)*kk2+(-co47+co45-co30+co2
?    8)*ge*k2*kk1+(co48-co46+co31-co29)*ge*k2)*lk**2+((co48-co46+co3
   1-co29)*k1*k2**2*kk1+(-co47-co45-co30-co28)*k1*k2**2)*kk2+(-co5
1    5+co53-co40+co38)*k1*k2**2*kk1+(co54+co52+co39+co37)*k1*k2**2)*
2    ll

 me(1,5) = (((co68-co75)*k1*kk1+(co72+co63)*k1)*lk*ll*p2+((co62-co7
1    1)*ge*kk1+(co74-co67)*ge)*lk**3*ll)*xz+(((-co72-co63)*k1**2*kk1
2    +(co75+co68)*k1**2)*lk*p2+((co67-co74)*ge*k1*kk1+(co71+co62)*ge
3    *k1)*lk**3)*xy+((co72-co63)*kk1-co75+co68)*la*lk*p2
```

```
 me(1,6) = (((3*co19-3*co21)*k1*kk1+(3*co20+3*co18)*k1)*lk*l1*p2+((
1    co7-co9)*ge*kk1+(co10-co8)*ge)*lk**3*l1)*xz+(((-3*co20-3*co18)*
2    k1**2*kk1+(3*co21+3*co19)*k1**2)*lk*p2+((co8-co10)*ge*k1*kk1+(c
3    o9+co7)*ge*k1)*lk**3)*xy+((3*co20-3*co18)*kk1-3*co21+3*co19)*la
4    *lk*p2

 me(1,7) = (((co68-co75)*ge*k1*kk1+(co72+co63)*ge*k1)*lk*p2+((co62-
1    co71)*ge*kk1+(co74-co67)*ge)*lk*l1**2+((co71+co62)*k1**2*kk1+(-
2    co74-co67)*k1**2)*lk)*xz+((co67-co74)*ge*k1*kk1+(co71+co62)*ge*
3    k1)*lk*l1*xy+((co63-co72)*ge*kk1+(co75-co68)*ge)*lk*l1*p2+((co7
4    4-co67)*k1*kk1+(-co71-co62)*k1)*lk*l1

 me(1,8) = (((3*co19-3*co21)*ge*k1*kk1+(3*co20+3*co18)*ge*k1)*lk*p2
1    +((co7-co9)*ge*kk1+(co10-co8)*ge)*lk*l1**2+((co9+co7)*k1**2*kk1
2    +(-co8-co10)*k1**2)*lk)*xz+((co8-co10)*ge*k1*kk1+(co9+co7)*ge*k
3    1)*lk*l1*xy+((3*co18-3*co20)*ge*kk1+(3*co21-3*co19)*ge)*lk*l1*p
4    2+((co10-co8)*k1*kk1+(-co9-co7)*k1)*lk*l1

 me(2,2) = (((2*co44-2*co35-2*co27+2*co25)*ge*k2*kk2**2+(-2*co51-2*
1    co43+4*co36-2*co34+2*co24)*ge*k2*kk2+(2*co44-2*co35+2*co27-2*co
2    25)*ge*k2)*lk**2+(2*co44-2*co35+2*co27-2*co25)*k2**3*kk2**2+(-2
3    *co51-2*co43+2*co34-4*co26-2*co24)*k2**3*kk2+(2*co44+2*co35+2*c
4    o27+2*co25)*k2**3)*l1*xz+(((co51-2*co36+co34)*ge*k2**2*kk2**2+(
5    -2*co44+2*co35-2*co27+2*co25)*ge*k2**2*kk2+(co43+2*co26+co24)*g
6    e*k2**2)*lk**2+(co43+2*co26+co24)*k2**4*kk2**2+(-2*co44-2*co35-
7    2*co27-2*co25)*k2**4*kk2+(co51+2*co36+co34)*k2**4)*xy+(((co43-2
8    *co26+co24)*ge*kk2**2+(-2*co44+2*co35+2*co27-2*co25)*ge*kk2+(co
9    51-2*co36+co34)*ge)*lk**2+(co51-2*co36+co34)*k2**2*kk2**2+(-2*c
:    o44+2*co35-2*co27+2*co25)*k2**2*kk2+(co43+2*co26+co24)*k2**2)*l
;    1**2+((co43-2*co26+co24)*kk2**2+(-2*co44+2*co35+2*co27-2*co25)*
<    kk2+co51-2*co36+co34)*la*lk**2

 me(2,3) = (((((-co47+co45+co30-co28)*ge*kk1+(co48-co46-co31+co29)*
1    ge)*kk2+(co54-co52-co39+co37)*ge*kk1+(-co55+co53+co40-co38)*ge)
2    *lk**2+((-co55+co53+co40-co38)*k1*k2*kk1+(co54+co52-co39-co37)*
3    k1*k2)*kk2+(co48-co46+co31-co29)*k1*k2*kk1+(-co47-co45-co30-co2
4    8)*k1*k2)*l1**2+(((co55-co53-co40+co38)*ge*k1*k2*kk1+(-co54-co5
5    2+co39+co37)*ge*k1*k2)*kk2+(-co48+co46-co31+co29)*ge*k1*k2*kk1+
6    (co47+co45+co30+co28)*ge*k1*k2)*lk**2+((co47+co45+co30+co28)*k1
7    **2*k2**2*kk1+(-co48-co46-co31-co29)*k1**2*k2**2)*kk2+(-co54-co
8    52-co39-co37)*k1**2*k2**2*kk1+(co55+co53+co40+co38)*k1**2*k2**2
9    )*xz+((((-co54+co52+co39-co37)*ge*k2*kk1+(co55-co53-co40+co38)*
:    ge*k2)*kk2+(co47-co45+co30-co28)*ge*k2*kk1+(-co48+co46-co31+co2
;    9)*ge*k2)*lk**2+((-co48+co46-co31+co29)*k1*k2**2*kk1+(co47+co45
<    +co30+co28)*k1*k2**2)*kk2+(co55-co53+co40-co38)*k1*k2**2*kk1+(-
```

```
=    co54-co52-co39-co37)*k1*k2**2)*l1*xy+((((co48-co46-co31+co29)*g
>    e*k1*kk1+(-co47-co45+co30+co28)*ge*k1)*kk2+(-co55+co53+co40-co3
?    8)*ge*k1*kk1+(co54+co52-co39-co37)*ge*k1)*lk**2+((co54+co52-co3
    9-co37)*k1**2*k2*kk1+(-co55-co53+co40+co38)*k1**2*k2)*kk2+(-co4
1    7-co45-co30-co28)*k1**2*k2*kk1+(co48+co46+co31+co29)*k1**2*k2)*
2    l1

 me(2,4) = ((((-co43+2*co26-co24)*ge*kk2**2+(2*co44-2*co35-2*co27+2
1    *co25)*ge*kk2+(-co51+2*co36-co34)*ge)*lk**2+(-co51+2*co36-co34)
2    *k2**2*kk2**2+(2*co44-2*co35+2*co27-2*co25)*k2**2*kk2+(-co43-2*
3    co26-co24)*k2**2)*l1**2+((co51-2*co36+co34)*ge*k2**2*kk2**2+(-2
4    *co44+2*co35-2*co27+2*co25)*ge*k2**2*kk2+(co43+2*co26+co24)*ge*
5    k2**2)*lk**2+(co43+2*co26+co24)*k2**4*kk2**2+(-2*co44-2*co35-2*
6    co27-2*co25)*k2**4*kk2+(co51+2*co36+co34)*k2**4)*xz+(((-co44+co
7    35+co27-co25)*ge*k2*kk2**2+(co51+co43-2*co36+co34-co24)*ge*k2*k
8    k2+(-co44+co35-co27+co25)*ge*k2)*lk**2+(-co44+co35-co27+co25)*k
9    2**3*kk2**2+(co51+co43-co34+2*co26+co24)*k2**3*kk2+(-co44-co35-
:    co27-co25)*k2**3)*l1*xy+(((co44-co35-co27+co25)*ge*k2*kk2**2+(-
;    co51-co43+2*co36-co34+co24)*ge*k2*kk2+(co44-co35+co27-co25)*ge*
<    k2)*lk**2+(co44-co35+co27-co25)*k2**3*kk2**2+(-co51-co43+co34-2
=    *co26-co24)*k2**3*kk2+(co44+co35+co27+co25)*k2**3)*l1

 me(2,5) = (((co42-co57)*k2*kk2+(co50+co33)*k2)*lk*l1*p2+((co32-co4
1    9)*ge*kk2+(co56-co41)*ge)*lk**3*l1)*xz+(((-co50-co33)*k2**2*kk2
2    +(co57+co42)*k2**2)*lk*p2+((co41-co56)*ge*k2*kk2+(co49+co32)*ge
3    *k2)*lk**3)*xy+((co50-co33)*kk2-co57+co42)*la*lk*p2

 me(2,6) = (((3*co15-3*co17)*k2*kk2+(3*co16+3*co14)*k2)*lk*l1*p2+((
1    co3-co5)*ge*kk2+(co6-co4)*ge)*lk**3*l1)*xz+(((-3*co16-3*co14)*k
2    2**2*kk2+(3*co17+3*co15)*k2**2)*lk*p2+((co4-co6)*ge*k2*kk2+(co5
3    +co3)*ge*k2)*lk**3)*xy+((3*co16-3*co14)*kk2-3*co17+3*co15)*la*l
4    k*p2

 me(2,7) = (((co42-co57)*ge*k2*kk2+(co50+co33)*ge*k2)*lk*p2+((co32-
1    co49)*ge*kk2+(co56-co41)*ge)*lk*l1**2+((co49+co32)*k2**2*kk2+(-
2    co56-co41)*k2**2)*lk)*xz+((co41-co56)*ge*k2*kk2+(co49+co32)*ge*
3    k2)*lk*l1*xy+((co33-co50)*ge*kk2+(co57-co42)*ge)*lk*l1*p2+((co5
4    6-co41)*k2*kk2+(-co49-co32)*k2)*lk*l1

 me(2,8) = (((3*co15-3*co17)*ge*k2*kk2+(3*co16+3*co14)*ge*k2)*lk*p2
1    +((co3-co5)*ge*kk2+(co6-co4)*ge)*lk*l1**2+((co5+co3)*k2**2*kk2+
2    (-co6-co4)*k2**2)*lk)*xz+((co4-co6)*ge*k2*kk2+(co5+co3)*ge*k2)*
3    lk*l1*xy+((3*co14-3*co16)*ge*kk2+(3*co17-3*co15)*ge)*lk*l1*p2+(
4    (co6-co4)*k2*kk2+(-co5-co3)*k2)*lk*l1
```

```
me(3,3) = (((-2*co70+2*co65+2*co61-2*co59)*ge*k1*kk1**2+(2*co73+2*
1    co69-4*co66+2*co64-2*co58)*ge*k1*kk1+(-2*co70+2*co65-2*co61+2*c
2    o59)*ge*k1)*lk**2+(-2*co70+2*co65-2*co61+2*co59)*k1**3*kk1**2+(
3    2*co73+2*co69-2*co64+4*co60+2*co58)*k1**3*kk1+(-2*co70-2*co65-2
4    *co61-2*co59)*k1**3)*l1*xz+(((co69-2*co60+co58)*ge*kk1**2+(-2*c
5    o70+2*co65+2*co61-2*co59)*ge*kk1+(co73-2*co66+co64)*ge)*lk**2+(
6    co73-2*co66+co64)*k1**2*kk1**2+(-2*co70+2*co65-2*co61+2*co59)*k
7    1**2*kk1+(co69+2*co60+co58)*k1**2)*l1**2*xy+((co73-2*co66+co64)
8    *ge*k1**2*kk1**2+(-2*co70+2*co65-2*co61+2*co59)*ge*k1**2*kk1+(c
9    o69+2*co60+co58)*ge*k1**2)*lk**2+(co69+2*co60+co58)*k1**4*kk1**
:    2+(-2*co70-2*co65-2*co61-2*co59)*k1**4*kk1+(co73+2*co66+co64)*k
;    1**4

me(3,4) = (((((-co54+co52+co39-co37)*ge*k2+(-co48+co46+co31-co29)*
1    ge*k1)*kk1+(co55-co53-co40+co38)*ge*k2+(co47+co45-co30-co28)*ge
2    *k1)*kk2+((co47-co45+co30-co28)*ge*k2+(co55-co53-co40+co38)*ge*
3    k1)*kk1+(-co48+co46-co31+co29)*ge*k2+(-co54-co52+co39+co37)*ge*
4    k1)*lk**2+(((-co48+co46-co31+co29)*k1*k2**2+(-co54-co52+co39+co
5    37)*k1**2*k2)*kk1+(co47+co45+co30+co28)*k1*k2**2+(co55+co53-co4
6    0-co38)*k1**2*k2)*kk2+((co55-co53+co40-co38)*k1*k2**2+(co47+co4
7    5+co30+co28)*k1**2*k2)*kk1+(-co54-co52-co39-co37)*k1*k2**2+(-co
8    48-co46-co31-co29)*k1**2*k2)*l1*xz+((((co47-co45-co30+co28)*ge*
9    kk1+(-co48+co46+co31-co29)*ge)*kk2+(-co54+co52+co39-co37)*ge*kk
:    1+(co55-co53-co40+co38)*ge)*lk**2+((co55-co53-co40+co38)*k1*k2*
;    kk1+(-co54-co52+co39+co37)*k1*k2)*kk2+(-co48+co46-co31+co29)*k1
<    *k2*kk1+(co47+co45+co30+co28)*k1*k2)*l1**2*xy+(((co55-co53-co40
=    +co38)*ge*k1*k2*kk1+(-co54-co52+co39+co37)*ge*k1*k2)*kk2+(-co48
>    +co46-co31+co29)*ge*k1*k2*kk1+(co47+co45+co30+co28)*ge*k1*k2)*l
?    k**2+((co47+co45+co30+co28)*k1**2*k2**2*kk1+(-co48-co46-co31-co
   29)*k1**2*k2**2)*kk2+(-co54-co52-co39-co37)*k1**2*k2**2*kk1+(co
1    55+co53+co40+co38)*k1**2*k2**2

me(3,5) = (((-co72-co63)*k1**2*kk1+(co75+co68)*k1**2)*lk*p2+((co67
1    -co74)*ge*k1*kk1+(co71+co62)*ge*k1)*lk**3)*xz+(((co75-co68)*k1*
2    kk1+(-co72-co63)*k1)*lk*l1*p2+((co71-co62)*ge*kk1+(co67-co74)*g
3    e)*lk**3*l1)*xy

me(3,6) = (((-3*co20-3*co18)*k1**2*kk1+(3*co21+3*co19)*k1**2)*lk*p
1    2+((co8-co10)*ge*k1*kk1+(co9+co7)*ge*k1)*lk**3)*xz+(((3*co21-3*
2    co19)*k1*kk1+(-3*co20-3*co18)*k1)*lk*l1*p2+((co9-co7)*ge*kk1+(c
3    o8-co10)*ge)*lk**3*l1)*xy

me(3,7) = (((co72-co63)*ge*kk1+(co68-co75)*ge)*lk*l1*p2+(((co67-co
1    74)*ge-co74+co67)*k1*kk1+((co71+co62)*ge+co71+co62)*k1)*lk*l1)*
2    xz+((co71-co62)*ge*kk1+(co67-co74)*ge)*lk*l1**2*xy+((co68-co75)
```

```
3    *ge*k1*kk1+(co72+co63)*ge*k1)*lk*p2+((co71+co62)*k1**2*kk1+(-co
4    74-co67)*k1**2)*lk

 me(3,8) = (((3*co20-3*co18)*ge*kk1+(3*co19-3*co21)*ge)*lk*ll*p2+((
1    (co8-co10)*ge+co8-co10)*k1*kk1+((co9+co7)*ge+co9+co7)*k1)*lk*ll
2    )*xz+((co9-co7)*ge*kk1+(co8-co10)*ge)*lk*ll**2*xy+((3*co19-3*co
3    21)*ge*k1*kk1+(3*co20+3*co18)*ge*k1)*lk*p2+((co9+co7)*k1**2*kk1
4    +(-co8-co10)*k1**2)*lk

 me(4,4) = (((-2*co44+2*co35+2*co27-2*co25)*ge*k2*kk2**2+(2*co51+2*
1    co43-4*co36+2*co34-2*co24)*ge*k2*kk2+(-2*co44+2*co35-2*co27+2*c
2    o25)*ge*k2)*lk**2+(-2*co44+2*co35-2*co27+2*co25)*k2**3*kk2**2+(
3    2*co51+2*co43-2*co34+4*co26+2*co24)*k2**3*kk2+(-2*co44-2*co35-2
4    *co27-2*co25)*k2**3)*ll*xz+(((co43-2*co26+co24)*ge*kk2**2+(-2*c
5    o44+2*co35+2*co27-2*co25)*ge*kk2+(co51-2*co36+co34)*ge)*lk**2+(
6    co51-2*co36+co34)*k2**2*kk2**2+(-2*co44+2*co35-2*co27+2*co25)*k
7    2**2*kk2+(co43+2*co26+co24)*k2**2)*ll**2*xy+((co51-2*co36+co34)
8    *ge*k2**2*kk2**2+(-2*co44+2*co35-2*co27+2*co25)*ge*k2**2*kk2+(c
9    o43+2*co26+co24)*ge*k2**2)*lk**2+(co43+2*co26+co24)*k2**4*kk2**
:    2+(-2*co44-2*co35-2*co27-2*co25)*k2**4*kk2+(co51+2*co36+co34)*k
;    2**4

 me(4,5) = (((-co50-co33)*k2**2*kk2+(co57+co42)*k2**2)*lk*p2+((co41
1    -co56)*ge*k2*kk2+(co49+co32)*ge*k2)*lk**3)*xz+(((co57-co42)*k2*
2    kk2+(-co50-co33)*k2)*lk*ll*p2+((co49-co32)*ge*kk2+(co41-co56)*g
3    e)*lk**3*ll)*xy

 me(4,6) = (((-3*co16-3*co14)*k2**2*kk2+(3*co17+3*co15)*k2**2)*lk*p
1    2+((co4-co6)*ge*k2*kk2+(co5+co3)*ge*k2)*lk**3)*xz+(((3*co17-3*c
2    o15)*k2*kk2+(-3*co16-3*co14)*k2)*lk*ll*p2+((co5-co3)*ge*kk2+(co
3    4-co6)*ge)*lk**3*ll)*xy

 me(4,7) = (((co50-co33)*ge*kk2+(co42-co57)*ge)*lk*ll*p2+(((co41-co
1    56)*ge-co56+co41)*k2*kk2+((co49+co32)*ge+co49+co32)*k2)*lk*ll)*
2    xz+((co49-co32)*ge*kk2+(co41-co56)*ge)*lk*ll**2*xy+((co42-co57)
3    *ge*k2*kk2+(co50+co33)*ge*k2)*lk*p2+((co49+co32)*k2**2*kk2+(-co
4    56-co41)*k2**2)*lk

 me(4,8) = (((3*co16-3*co14)*ge*kk2+(3*co15-3*co17)*ge)*lk*ll*p2+((
1    (co4-co6)*ge-co6+co4)*k2*kk2+((co5+co3)*ge+co5+co3)*k2)*lk*ll)*
2    xz+((co5-co3)*ge*kk2+(co4-co6)*ge)*lk*ll**2*xy+((3*co15-3*co17)
3    *ge*k2*kk2+(3*co16+3*co14)*ge*k2)*lk*p2+((co5+co3)*k2**2*kk2+(-
4    co6-co4)*k2**2)*lk

 me(5,5) = (co78*lk**2*p2**2+co76*ge*lk**4)*xy+co78*la*p2**2
```

```
me(5,6) = (3*co23*lk**2*p2**2+co11*ge*lk**4)*xy+3*co23*la*p2**2

me(5,7) = (co77*ge-co77)*lk**2*p2*xz+co76*ge*lk**2*ll*xy

me(5,8) = (3*co22*ge-co12)*lk**2*p2*xz+co11*ge*lk**2*ll*xy

me(6,6) = (9*co13*lk**2*p2**2+co1*ge*lk**4)*xy+9*co13*la*p2**2

me(6,7) = (co12*ge-3*co22)*lk**2*p2*xz+co11*ge*lk**2*ll*xy

me(6,8) = (3*co2*ge-3*co2)*lk**2*p2*xz+co1*ge*lk**2*ll*xy

me(7,7) = -lmda*(-co76*xy-co76)+(co78*ge*p2**2-co76*lmda+co76*ge*l
1    l**2)*xy+co78*ge*p2**2-co76*lmda+co76*ge*ll**2+co76*lk**2

me(7,8) = -lmda*(-co11*xy-co11)+(3*co23*ge*p2**2-co11*lmda+co11*ge
1    *ll**2)*xy+3*co23*ge*p2**2-co11*lmda+co11*ge*ll**2+co11*lk**2

me(8,8) = -lmda*(-co1*xy-co1)+(9*co13*ge*p2**2-co1*lmda+c   ge*ll*
1    *2)*xy+9*co13*ge*p2**2-co1*lmda+co1*ge*ll**2+co1*lk**2
```

D.2 Matrix Mf

$$\mathbf{M}_f(i,j) = \mathbf{M}_f(j,i) \quad for \quad i \neq j \quad (i,j = 1,\dots,8)$$

```
mf(1,1) = -((-co69+2*co60-co58)*kk1**2+(2*co70-2*co65-2*co61+2*co5
1    9)*kk1-co73+2*co66-co64)*la

mf(1,2) = -(((-co47+co45+co30-co28)*kk1+co48-co46-co31+co29)*kk2+(
1    co54-co52-co39+co37)*kk1-co55+co53+co40-co38)*la

mf(1,3) = 0

mf(1,4) = 0

mf(1,5) = 0

mf(1,6) = 0

mf(1,7) = 0

mf(1,8) = 0
```

```
mf(2,2) = -((-co43+2*co26-co24)*kk2**2+(2*co44-2*co35-2*co27+2*co2
1    5)*kk2-co51+2*co36-co34)*la

mf(2,3) = 0

mf(2,4) = 0

mf(2,5) = 0

mf(2,6) = 0

mf(2,7) = 0

mf(2,8) = 0

mf(3,3) = -((-co69+2*co60-co58)*kk1**2+(2*co70-2*co65-2*co61+2*co5
1    9)*kk1-co73+2*co66-co64)*la

mf(3,4) = -(((-co47+co45+co30-co28)*kk1+co48-co46-co31+co29)*kk2+(
1    co54-co52-co39+co37)*kk1-co55+co53+co40-co38)*la

mf(3,5) = 0

mf(3,6) = 0

mf(3,7) = 0

mf(3,8) = 0

mf(4,4) = -((-co43+2*co26-co24)*kk2**2+(2*co44-2*co35-2*co27+2*co2
1    5)*kk2-co51+2*co36-co34)*la

mf(4,5) = 0

mf(4,6) = 0

mf(4,7) = 0

mf(4,8) = 0

mf(5,5) = co76*la

mf(5,6) = co11*la

mf(5,7) = 0
```

```
mf(5,8) = 0

mf(6,6) = co1*la

mf(6,7) = 0

mf(6,8) = 0

mf(7,7) = co76*xy+co76

mf(7,8) = co11*xy+co11

mf(8,8) = co1*xy+co1
```

D.3 Integrated Coefficients

The integrated coefficients, co1 to co78, shown in matrices M_e and M_f are the results of the product of functions ξ and η integrated from $z = 0$ to $z = l$.

Functions ξ and η as well as the relations between co$n$ $(n = 1.....78)$ and functions ξ, η are given in table D-1.

From table D-1, we arrive at the following values.

```
kk1=0.7340955137589
kk2=1.018467318759
k1=1.875104068712
k2=4.694091132974
p2=0.5*pi

co1=0.5
co2=0.1061032953945969
co3=0.4989995345474672
co4=0.1108751276626285
co5=- 5.798956073142442
co6=- 5.691469236183133
co7=0.03006018182478657
```

ξ	η											
	sin(t1)	cos(t1)	sin(t2)	cos(t2)	sinh(t2)	cosh(t2)	sin(t3)	cos(t3)	sinh(t3)	cosh(t3)	sin(t4)	cos(t4)
sin(t1)	co1											
cos(t1)	co2	co13										
sin(t2)	co3	co14	co24									
cos(t2)	co4	co15	co25	co34								
sinh(t2)	co5	co16	co26	co35	co43							
cosh(t2)	co6	co17	co27	co36	co44	co51						
sin(t3)	co7	co18	co28	co37	co45	co52	co58					
cos(t3)	co8	co19	co29	co38	co46	co53	co59	co64				
sinh(t3)	co9	co20	co30	co39	co47	co54	co60	co65	co69			
cosh(t3)	co10	co21	co31	co40	co48	co55	co61	co66	co70	co73		
sin(t4)	co11	co22	co32	co41	co49	co56	co62	co67	co71	co74	co76	
cos(t4)	co12	co23	co33	co42	co50	co57	co63	co68	co72	co75	co77	co78

Note: t1=3 p2 z / l   t2=k2 z / l

t3=k1 z / l   t4=p2 z / l

p2 : 0.5 π   k1 : 1.8751   k2 : 4.694

Table D-1: Functions ξ and η
(Clamped-Free, Normal Mode Functions)

co8=0.3478402254263256
co9=- 0.2432880298097214
co10=- 0.04890990761714533
co11=0
co12=0.3183098861837907
co13=0.5
co14=0.101726459217963
co15=0.5009446645167207
co16=- 5.926688707716224
co17=- 5.821560750035544
co18=- 0.3408655849552808
co19=0.07554528409550446
co20=- 0.6562209160646115
co21=- 0.6114155741352192
co22=- 0.3183098861837907
co23=0
co24=0.4980514051656076
co25=0.1064812333022709
co26=- 5.714131203766796
co27=- 5.606622101392816
co28=0.03475931115663723
co29=0.3486792690147293
co30=- 0.2341827624244058
co31=- 0.03869781747047025
co32=0.004389358274729287
co33=0.3201612686560902
co34=0.5019485948343924
co35=- 6.032689670886772
co36=- 5.927129326459122
co37=- 0.3424954405873828
co38=0.07770619523077635
co39=- 0.6627446803520218
co40=- 0.6173170841492832
co41=- 0.3201345481538806
co42=0.00146882275175669
co43=317.6223970917853
co44=318.0691431037289
co45=10.78126487405295
co46=0.6337238477409987
co47=25.64535716383292
co48=28.36465756278566
co49=10.47070965266181
co50=3.311674398369181
co51=318.6223970917853
co52=10.85325055404595

```
co53=0.8185862051268238
co54=25.73613685358632
co55=28.60744933975992
co56=10.53306602928041
co57=3.503841701282994
co58=0.5762267152276445
co59=0.2427119121868963
co60=1.103441151283322
co58=0.5762267152276445
co59=0.2427119121868963
co60=1.103441151283322
co61=1.343334932063346
co62=0.5357954242210862
co63=0.3590249035362438
co64=0.4237732847723555
co65=0.276727465644721
co66=0.5946402690834619
co67=0.2080415754963285
co68=0.4488420127305808
co69=2.333604078798335
co70=2.703413018359166
co71=1.045891914197238
co72=0.5225168922652508
co73=3.333604078798335
co74=1.260362987158822
co75=0.876155731545073
co76=0.5
co77=0.3183098861837907
co78=0.5
```

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